Parity Breaking Bifurcation in Inhomogeneous Systems

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Parity breaking instabilities of spatially periodic patterns are considered. In homogeneous systems such instabilities produce steadily drifting patterns. Spatial inhomogeneities are shown to lead to pattern pinning. The transition from pinned patterns to drifting ones may be surprisingly complex. Examples are described containing infinite cascades of global bifurcations. The values of the bifurcation parameter at which these occur obey a simple scaling law. The predicted dynamics provide a qualitative understanding of recent experiments on binary fluid convection in an annulus.

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Primary instabilities in nonequilibrium systems such as Rayleigh-Bénard convection [1] or the Mullins-Sekerka instability of a moving interface between two phases [2] typically lead to spatial patterns, i.e., to convective rolls or cellular interfaces. Most theoretical approaches to describing such instabilities rely on symmetry considerations. The system is assumed to be spatially translation and reflection invariant, and the first instability is interpreted as a symmetry breaking bifurcation in which the continuous translation invariance of the basic state is broken to a discrete one but the reflection invariance is retained [3]. When the instability parameter (e.g., the Rayleigh number) is raised farther, the periodic pattern may itself become unstable in a secondary symmetry breaking bifurcation in which the reflection invariance is broken. Such secondary instabilities produce a nonsymmetric pattern that travels either to the left or to the right, depending on its asymmetry. Such bifurcations have been observed in a number of recent experiments [4].

With periodic boundary conditions the normal form describing this secondary instability is given by [5]

$$\dot{c} = (\mu - c^2)\, c, \qquad \dot{\phi} = c\,, \tag{1}$$

where c is the phase velocity of the bifurcating traveling wave, ϕ is its spatial phase, and μ is the deviation of the basic parameter from its value at the secondary instability. The equation for c is a supercritical pitchfork so that for $\mu < 0$ there is only the symmetric steady state $c = 0$ while for $\mu > 0$ there is a pair of nonsymmetric solutions corresponding to waves propagating in opposite directions with speed $c = \pm\sqrt{\mu}$.

Perfect translation invariance is rare, however, and spatial inhomogeneities (or distant sidewalls) may alter the above picture and introduce qualitatively new behavior. That this is likely is indicated by the fact that pure traveling waves can only exist in systems with perfect translation invariance. In this Letter we propose a two-dimensional normal form that describes the consequences of broken translation invariance on the secondary symmetry breaking instability of a periodic, reflection-invariant state and describe some of its dynamical properties.

The proposed normal form is a perturbation of (1) such that the reflection symmetry $(c, \phi) \rightarrow (-c, -\phi)$ is preserved but the translation invariance $\phi \rightarrow \phi + \theta$ is broken. With periodic boundary conditions we obtain to leading order [6]

$$\dot{c} = (\mu + \delta\cos\phi - c^2)\, c + \eta\sin\phi\,, \tag{2a}$$

$$\dot{\phi} = c - \epsilon\sin\phi\,, \tag{2b}$$

where (δ, η, ϵ) are parameters proportional to the amplitude of the spatial inhomogeneity, assumed to be small and with spatial period 2π. A systematic derivation of (2) will be given elsewhere [6], and shows that in general the sines and cosines have to be replaced by arbitrary odd and even 2π-periodic functions, i.e., in writing (2) we have only retained the leading terms in the Fourier expansion of these functions.

The breaking of the translation invariance is responsible for the appearance of two fixed points of the form $(c, \phi) = (0, 0), (0, \pi)$ corresponding to steady rolls with a preferred phase. In the following we refer to these as pinned states and denote them by $SS_{0,\pi}$. These reflection-symmetric solutions are always present and take the place of the circle of solutions $(0, \phi), \phi \in [0, 2\pi)$, in the translation-invariant system. Additional fixed points of the form (c_0, ϕ_0) are sometimes present. Since ϕ is fixed, these solutions also represent steady rolls; however, because $c_0 \neq 0$ these rolls are no longer symmetric under reflection. At most two such nonsymmetric fixed points $P^{\pm}$ can be present. The propagating states are described by limit cycles in the system (2). There are in general three types of such limit cycles, those that are born in Hopf bifurcations from $SS_{0,\pi}$ or from $P^{\pm}$ and are characterized by oscillations in the phase ϕ, and the rotation waves (RW) for which ϕ increases monotonically with time. The former thus describe a symmetric or nonsymmetric roll pattern that alternately moves left and right, while the latter describe patterns that continually propagate in one or the other direction. Depending on parameters several such oscillations can coexist, often stably.

As the parameters are varied both the fixed points and limit cycles undergo a rich assortment of bifurcations [6]. Here we focus on just one case, $\delta = 0.3$, $\eta = 0.02$, $\epsilon = 0.2$, and describe (in part) the bifurcations that take place as μ is increased. At small μ we expect the effect of spatial inhomogeneities to dominate and hence the presence of steady or pinned states; only for larger μ does one expect a transition to a RW characteristic of the translation-invariant system. Even though the system (2) is two dimensional the sequence of (global) bifurcations that may occur in order to achieve a transition from a pinned state to a RW can be remarkably complex. In fact, for these parameter values the system (2) exhibits two infinite cascades of global bifurcations, as well as numerous others. The values of μ at which the global bifurcations form are found to obey a simple asymptotic relation and accumulate at a finite value. That such behavior is possible in two dimensions is known [7]; however, to our knowledge no explicit example of this behavior has been presented.

We now briefly describe what happens. For $\mu < -0.4$ the SS_0 solution is stable, while SS_π is a saddle. At $\mu = -0.4$ SS_0 undergoes a pitchfork bifurcation shedding two nonsymmetric fixed points $P^{\pm}$ which are stable. In the interval $-0.4 < \mu < 0.087\,35$ the two parts of the unstable manifold of SS_0, hereafter $W_0^{u\pm}$, spiral into $P^{\pm}$ [Fig. 1(a)]. With increasing μ the spiraling of $W_0^{u\pm}$ grows in amplitude and $W_0^{u\pm}$ approaches $W_\pi^{s\pm}$. At $\mu \approx 0.087\,35$ the first global bifurcation takes place, and $W_0^{u\pm}$ merges with $W_\pi^{s\pm}$, forming a heteroclinic connection between SS_0 and SS_π from above (below). See Fig. 1(b). With increasing μ this connection again breaks with the result that $W_0^{u\pm}$ now connects to $P^{\mp}$ [Fig. 1(c)]. As μ increases the spiraling of $W_0^{u\pm}$ grows, and a second global bifurcation at $\mu \approx 0.112$ produces a heteroclinic connection between SS_0 and SS_π [Fig. 1(d)], but this time from below (above). With further increase in μ this spiraling process repeats in ever smaller intervals of μ, repeatedly forming and breaking connections between

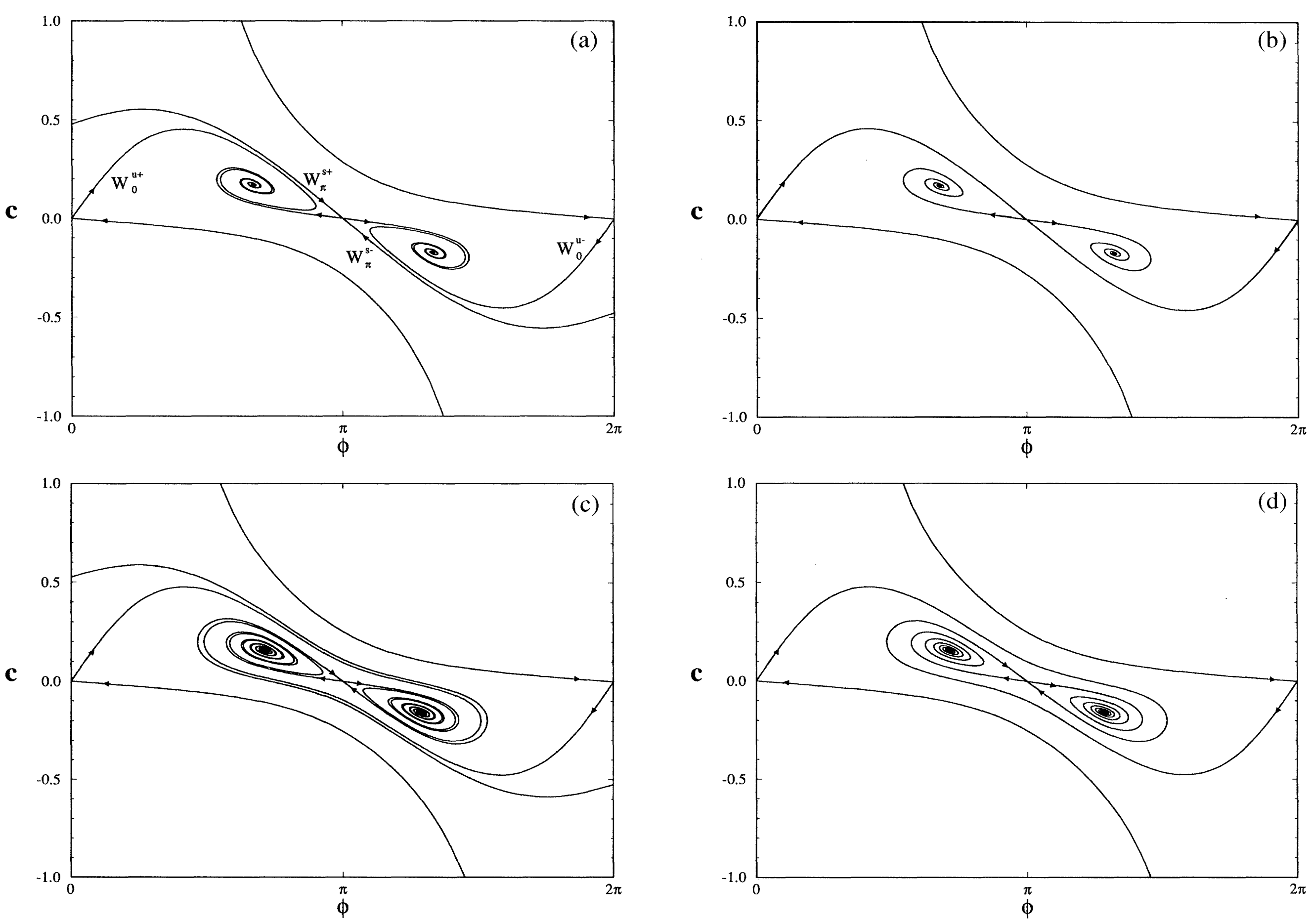

FIG. 1. The phase plane (ϕ, c) for (a) $\mu = 0.08$, (b) $\mu = 0.087\,35$, (c) $\mu = 0.11$, and (d) $\mu = 0.112$, showing the first two heteroclinic connections [(b) and (d)].

SS_0 and SS_π. As this happens the number of turns made by $W_0^{u\pm}$ around SS_π and its two satellite fixed points $P^\pm$ increases monotonically. The values μ_i at which the ith connection forms accumulate at $\mu_{SN} = 0.127\,630\,63$ at which point a pair of limit cycles (one stable, one unstable) are born, surrounding SS_π and $P^\pm$. At $\mu \approx 0.1283$ the inner (unstable) limit cycle forms a "figure eight" connection with SS_π and thereafter splits into two small limit cycles surrounding $P^\pm$ followed by the disappearance of the unstable oscillations in a subcritical Hopf bifurcation at $\mu \approx 0.131$, much as in the Takens-Bogdanov normal form with Z_2 symmetry [8]. Next, at $\mu \approx 0.164\,694$, two pairs of RW are born in saddle-node bifurcations, the ones with larger mean $|c|$ being stable [Fig. 2(a)]. At $\mu \approx 0.164\,938$ the unstable RW form a pair of homoclinic connections connecting SS_0 to itself and subsequently produce an unstable limit cycle surrounding the remaining stable limit cycle [Fig. 2(b)]. The latter is then destroyed in a second saddle-node bifurcation at $\mu_{SN} = 0.165\,264\,53$. As this value is approached from above an infinite sequence of global connections again takes place, all of which now coexist with the stable RW. Thus with increasing μ the complexity again unravels, leaving essentially the picture for $\mu < -0.4$ except that now both SS_0 and SS_π are unstable and a pair of stable RW is present.

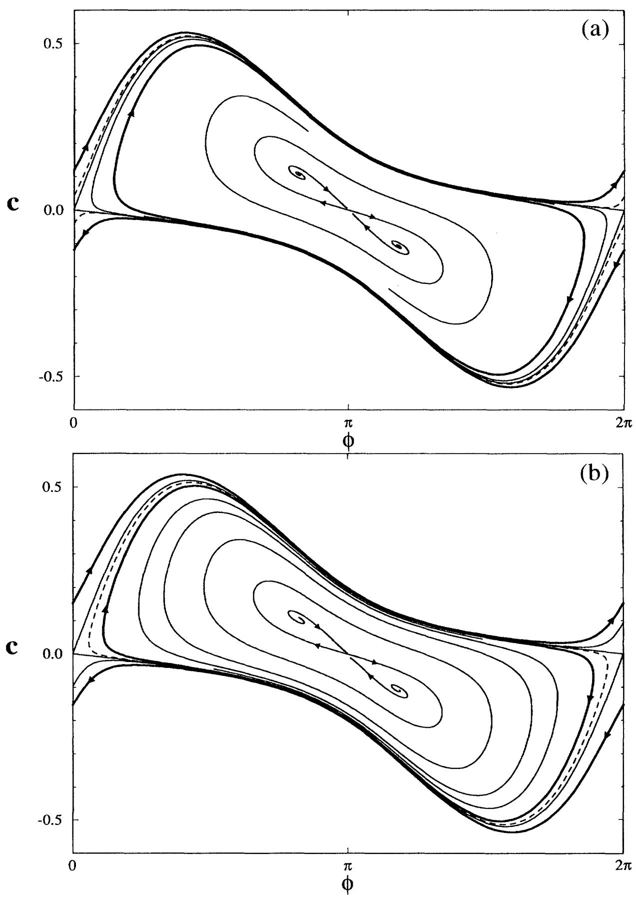

FIG. 2. Same as for Fig. 1 but for (a) $\mu = 0.1648$ and (b) $\mu = 0.1652$. Heavy (broken) lines indicate stable (unstable) periodic orbits.

Because the cascade of global bifurcations precedes the saddle-node bifurcation that produces the pair of limit cycles, we expect that the number $N(\mu)$ of connections formed by this value of μ is given by

$$N(\mu) \approx \frac{2K}{\sqrt{\mu_{SN} - \mu}}, \tag{3}$$

where K is a constant. The factor 2 is included since each rotation of W_0^u about SS_π and $P^\pm$ produces two global connections. We count the rotations by the intersections of W_0^u with the plane $\Sigma = \{(c, \phi) | \phi = \pi, c > 0\}$, and denote by μ_i the value of μ at which the ith connection forms. Thus, for large i,

$$\mu_{i+1} - \mu_i \approx \frac{1}{K}(\mu_{SN} - \mu_i)^{3/2}. \tag{4}$$

It follows that as $i \to \infty$, $\mu_i \to \mu_{SN}$, and an infinite number of connections *must* take place. We have checked these predictions by computing the number n $(= N/2)$ of intersections of W_0^u with Σ and plotting $1/n^2$ as a function of μ near μ_{SN}. As shown in Fig. 3 the result verifies the prediction (3), and can be used to deduce the values of μ_{SN} $(= 0.127\,630\,63)$ and of K $(= 0.200\,397\,9)$.

In order to describe the appearance of the solutions identified above it is helpful to represent them in the (x, t) plane using the representation [9]

$$\psi(x,t) = A \cos k[x + \phi(t)] + Bc(t) \sin k[x + \phi(t)], \tag{5}$$

where A, B are the amplitudes of the symmetric and nonsymmetric components of ψ, k is the wave number, and $c(t), \phi(t)$ solve (2). In Fig. 4 we show a stable limit cycle at $\mu = 0.1647$ coexisting with a stable rotating wave. The limit cycle corresponds to a direction-reversing wave

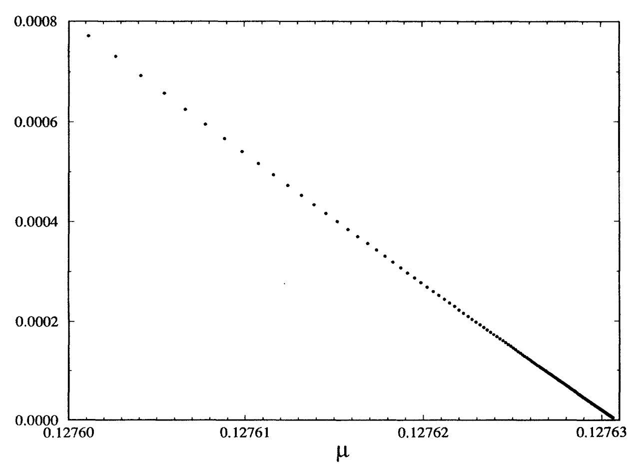

FIG. 3. Plot of $1/n^2$ against μ near $\mu = \mu_{SN}$.

(i.e., a wave that travels first to the left and then to the right, with zero net displacement), although the mechanism for its creation is quite different from that discussed in Ref. [10]. The RW propagate nonuniformly though always in the same direction, undergoing regular intervals of stasis followed by fast propagation. This behavior is typical of RW, the periods of stasis arising because the trajectory visits a neighborhood of the saddle point SS_0. With increasing μ the amplitude of the phase velocity modulation decreases and the solution approaches the translation-invariant result.

Although for other choices of parameters simpler scenarios may apply, the behavior we described is likely to be typical of the effects of spatial inhomogeneities on the bifurcation to traveling waves from a circle of steady states. As an application of the theory we mention the experiments of Ohlsen *et al.* [11,12] on binary fluid convection in an annular container. These authors find that the transition from traveling wave convection to steady convection is not sharp, even though three-dimensional effects are apparently absent. For larger μ they see the signature of the $\sqrt{\mu}$ dependence of the phase speed characteristic of this transition in a translation-invariant system. Near $\mu = 0$, however, this signature is washed out, and they find slowly traveling patterns even for $\mu < 0$. These patterns travel in fits and starts, with long periods of stasis separating propagation (cf. Fig. 7 of [12]), much as in Fig. 4(b). Moreover, when μ is large enough that $\phi \approx ct$, Eq. (2a) shows that the phase speed c satisfies the approximate equation

$$\dot{c} = (\mu - c^2)\,c + \eta \sin ct + \delta c \cos ct\,. \tag{6}$$

It follows that the phase speed will be modulated at leading order with *two* frequencies, namely, $\sqrt{\mu}$ and $2\sqrt{\mu}$. Modulation of the phase speed with just these two frequencies is observed in the experiments (cf. Fig. 4 of [12]). The present theory thus supports the interpretation of the experimental observations in terms of azimuthal inhomogeneities in the apparatus.

(a)

(b)

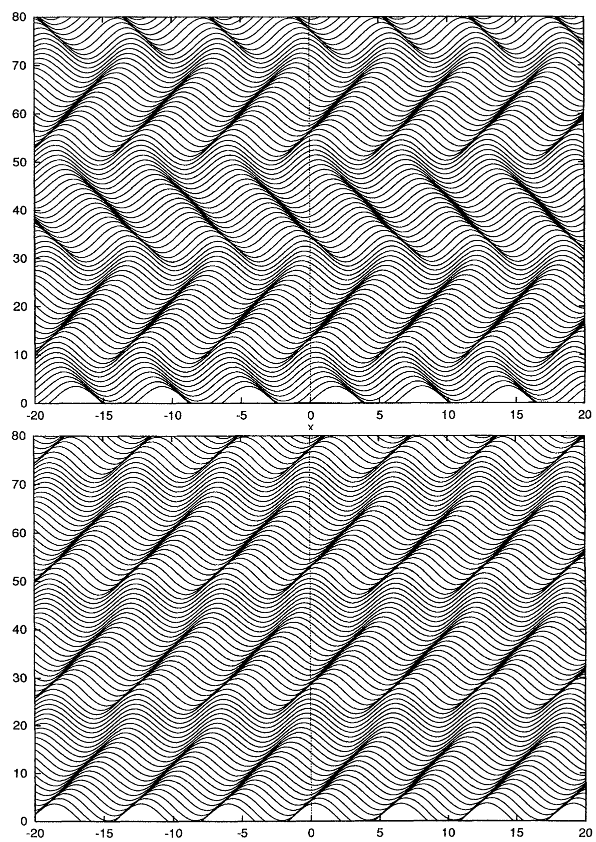

FIG. 4. Stable solutions in the (x, t) plane using the representation (5) with $A = 2.5$, $B = 2.5$, $k = 1.0$, showing (a) a symmetric oscillation, and (b) a rotating wave, both at $\mu = 0.1647$.

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