

Universal Shape of Scaling Functions in Turbulence

Janine Herweijer and Willem van de Water

Physics Department, Eindhoven University of Technology, P.O. Box 513, 5600 MB, Eindhoven, the Netherlands
(Received 21 March 1994; revised manuscript received 5 December 1994)

Using a novel device that enables the real-time measurement of high-order structure functions in turbulence with superior statistical accuracy, we study the shape of the structure function at scales where it is influenced by viscosity. The experiments on a variety of laboratory turbulent flows demonstrate that this shape is universal. We argue that it reflects a fundamental property of multifractals: multiscaling. We present accurate experimental scaling functions $\zeta(p)$ and, finally, comment on the nature of the skewness of the distribution function of velocity increments by comparing transverse and longitudinal distribution functions.

PACS numbers: 47.27.Gs, 02.50.-r, 05.45.+b

The existence of universal scaling behavior of the small-scale motion of turbulent flows is an exciting facet of the physics of turbulence. For large Reynolds numbers, a range of length scales $r \in [l, L]$ exists where there is a continuous flux of energy from large to small scales and moments of velocity fluctuations have a power law dependence on r with possibly universal exponents. In most experiments, the lower bound l of the inertial range $[l, L]$ turns out to be an order of magnitude larger than the smallest possible length scale η , where the local Reynolds number is of order 1 and viscous forces prevail.

An important instrument for quantifying scaling of turbulent velocity fluctuations is the structure function $G_p(r)$ of velocity differences $\Delta u(r)$, $G_p(r) = \langle \Delta u(r)^p \rangle_x$, with $\Delta u(r) = u(x+r) - u(x)$. The scaling of structure functions determines an exponent $\zeta(p)$ that is assumed to be universal at large Reynolds numbers. On the basis of dimensional arguments Kolmogorov [1] predicted $\zeta(p) = p/3$, but experimental results deviate from this prediction and show evidence for a nonlinear dependence of ζ on p [2].

The deviation from Kolmogorov's prediction is an expression of the intermittency of turbulent velocity fluctuations. The well-known multifractal model by Parisi and Frisch [3] is an attempt to capture intermittency in a geometrical framework. The idea is that velocity differences locally scale as $\Delta u(r) \approx r^h$, where the probability $P(h)$ of encountering an exponent h is $P(h) \approx r^{3-D(h)}$. A Legendre transformation connects the dimension $D(h)$ of the set of exponents h with the function $\zeta(p)$. In the multifractal framework, the "explanation" of a nonlinear scaling function $\zeta(p)$ is a nontrivial dimension function $D(h)$, i.e., a dimension function that does not collapse to a single point.

Multifractals are expected to exhibit multiscaling [4–6]. Multiscaling is the consequence of a singularity-dependent small-scale cutoff of the scaling region. In turbulent flows, the strongest singularities (smallest h) can survive to smaller scales before they are smoothed by viscosity. As structure functions of increasing order p probe the singularities of increasing strength, the scaling

behavior of $G_p(r)$ should extend to smaller r as p increases. The exact manner in which singularities are smoothed and the exact way in which the scaling more and more extends to viscous scales as p increases are still unresolved questions, but a crude model follows from equating the eddy turnover time $r/\Delta u(r)$ to the viscous momentum diffusion time r^2/ν , where ν is the kinematic viscosity [4]. Using this simple model, multiscaling is predicted to have a slight dependence on the Reynolds number. Therefore, in laboratory turbulent flows (that have modest values of the Reynolds number) the effect is expected to be small, and very careful experiments are needed. The search for experimental evidence of multiscaling, however, provides a highly desirable independent test of the multifractal model.

The scaling of the third-order longitudinal structure function $G_3(r) \approx r$ follows from the Navier-Stokes equation in the limit of zero viscosity [1]. Figure 1 shows $G_3(r)$ that was measured in a turbulent flow caused by the efflux of a jet in still air (details about the flow can be found in Table I). The result shows a clear inertial range

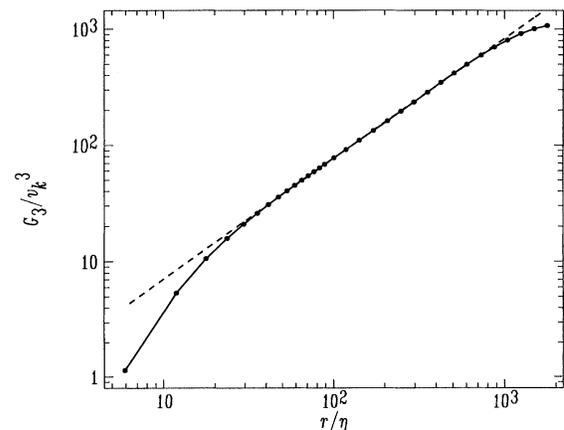


FIG. 1. Third-order structure function measured in jet turbulence with $Re_\lambda = 8.3 \times 10^2$ [11]. G_3 is normalized by the Kolmogorov velocity v_k ($= 0.17$ m/s).

where $G_3(r) \approx r^{\zeta(3)}$, with $\zeta(3) = 1.03$, which is slightly but significantly larger than 1. As Fig. 1 illustrates, the scaling behavior of structure functions $G_p(r)$ extends down to length scales $r \approx 30\eta$. For smaller distances, that are increasingly influenced by viscosity, G_p bends steeply down. At large length scales (here $r \approx 10^3\eta$), the finite size of the laboratory experiment is felt.

The central point of this Letter is that the shape of the structure function for $r \lesssim 30\eta$ exhibits a *universal* dependence on the order p . That is, a dependence that does not depend on the details of the experiment. We argue that this universal behavior is consistent with multiscaling.

The statistical accuracy of high-order structure functions is notoriously problematic. The problem is most severe at the smallest length scales (where our prime interest is). For increasing order p , $G_p(r)$ is an average over the increasingly rare instances of increasingly large velocity differences. We have built a special digital device that can measure high-order structure functions in real time through accumulated probability distribution functions of $\Delta u(r)$. Not only has this significantly eased the statistics problem, but the absence of stored time series and the immediate availability of the structure function has allowed systematic experimentation with the flow conditions [7]. The maximum number of velocity samples that we have taken is 1.5×10^9 , 2 orders of magnitude larger than what has been customary so far.

Briefly, the velocity signal from a hot wire [8] is passed through a four-pole antialiasing filter set at 10 kHz, or at 22.5 kHz, and digitized with a sampling time $\tau_s = 5 \times 10^{-5}$ s (2.2×10^{-5} s). The 12 bits of data are fed into a long (1024 positions) shift register and velocity differences $u(t) - u(t + \tau)$ are computed at discrete values of τ_i that are spaced exponentially on $[\tau_s, 1024\tau_s]$. In our experiments the size $\langle u^2 \rangle^{1/2}$ of the turbulent fluctuations is a small fraction of the mean velocity U , and we assume that a stationary probe cuts a line through a field of frozen turbulent fluctuations that is swept across it. Effectively, therefore, the probe measures space-dependent fluctuations $u(x) = u(Ut)$ [9].

Following a recent suggestion by Benzi *et al.* [10] we write the structure function as $G_p(r) = [f_p(r)r]^{\zeta(p)}$. The

function $f_p(r)$ gauges the deviation from ideal scaling behavior $G_p(r) \approx r^{\zeta(p)}$. As argued above, the multiscaling hypothesis predicts that this deviation in the intermediate viscous range will be smaller for larger moments p . For the moment we assume intermediate viscous scales to be $\eta \lesssim r \lesssim 30\eta$. In order to highlight the p dependence of $f_p(r)$ we define the function $g_p(r) \equiv f_p(r)/f_2(r)$. Roughly, the behavior of the function $g_p(r)$ for $p > 2$ will be as follows: (i) At very small scales, $r \approx \eta$, the velocity field is smooth and $G_p(r) \approx r^p$. Therefore, $g_p(r) \approx r^{\beta(p)}$, with a positive exponent $\beta(p) = p/\zeta(p) - 2/\zeta(2)$. (ii) In the intermediate viscous range the multiscaling hypothesis predicts $g_p(r) > 1$. (iii) For inertial range scales, $30\eta \lesssim r \lesssim L$, the (properly normalized) function $g_p(r)$ is unity because all scaling behavior has been captured in the exponent $\zeta(p)$. Summarizing, for r decreasing from $r \approx L$, the function g_p is first constant then increases and reaches a maximum at intermediate viscous scales, and decreases again when r becomes $O(\eta)$. According to the multiscaling hypothesis, the height of the maximum will be larger as the order p increases.

Figure 2 shows the function $g_p(r)$ on a log-log scale for various experiments. The Taylor microscale Reynolds numbers in these experiments ranged from $\text{Re}_\lambda = 3 \times 10^2$ to 8×10^2 . Especially the small Re_λ experiments have a small inertial range where $g_p(r)$ is flat. The spread of the measured $g_p(r)$ at large scales is not due to statistical error but reflects different macroscopic experimental conditions; those have been summarized in Table I and [11].

For the different flow conditions and the (small) range of Reynolds numbers that we have studied, Fig. 2 demonstrates the universality of the residual functions $g_p(r)$ for $r \lesssim 30\eta$. The p dependence of g_p is precisely as pre-

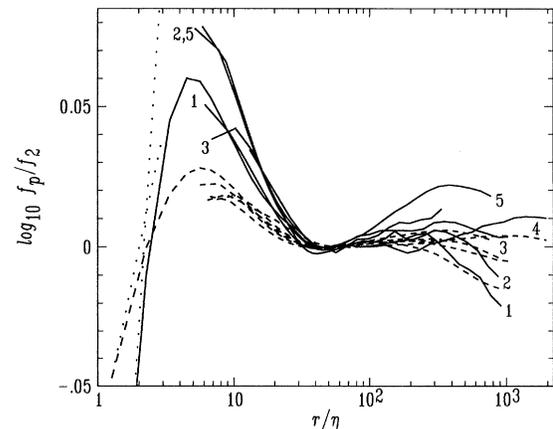


FIG. 2. Residual function $g_p(r)$ for $p = 4$ (dashed lines) and $p = 8$ (full lines) in a variety of turbulent flows. The numbers point to flow conditions in Table I and [11]. Experiment 1 was repeated at a large enough sampling rate to resolve $r \approx 2\eta$. The predicted small- r scaling behavior $g_p(r) \sim r^{\beta(p)}$ is shown by the dotted lines.

TABLE I. Characteristic parameters of the turbulent flows. The Taylor microscale Reynolds number Re_λ is defined with respect to a correlation length λ of turbulent fluctuations. L is the integral length scale of the flow that is determined from the large- r behavior of the correlation function. The flow conditions are summarized in [11].

Experiment	U (m/s)	u (m/s)	Re_λ	η (m)	L (m)
1	12.6	0.68	3.4×10^2	2.0×10^{-4}	0.095
2	11.0	0.94	5.3×10^2	1.9×10^{-4}	0.170
3	12.2	1.05	6.0×10^2	1.8×10^{-4}	0.130
4	12.5	2.27	8.1×10^2	9.5×10^{-5}	0.075
5	10.4	0.80	4.5×10^2	2.0×10^{-4}	0.170

dicted by the multiscaling hypothesis. For the curves in Fig. 2 with the filter set at 10 kHz the temporal resolution is not fine enough to observe the predicted maximum of $g_p(r)$. One of the experiments was also done at a large enough sampling rate to resolve $r \approx 2\eta$. The corresponding curves clearly show the maximum and agree with the predicted small- r scaling, $g_p(r) \sim r^{\beta(p)}$.

Structure functions such as shown in Fig. 1 have a clear scaling region that allows an unambiguous determination of a scaling exponent. Especially for laboratory flows at small Re_λ this is not always the case and measuring scaling exponents $\zeta(p)$ requires a normalization procedure [12]. In Fig. 3 we plot the scaling function $\zeta(p)/p$ for those of our experiments (2,4) that had a large enough Re_λ and displayed obvious scaling behavior. Therefore, there is no ambiguity as to the scaling interval and the value of the scaling exponent but we have vertically shifted curves such that $\zeta(3) = 1$. The actual values of $\zeta(3)$ varied from 0.96 to 1.03. These small but significant deviations from unity are important systematic errors whose origin we are currently trying to understand. By no means have moments of order 20 converged. We estimate that the largest converged moment is approximately 14. Still, at larger values of p we have detected clear scaling behavior.

The results of Fig. 3 clearly and firmly demonstrate the anomalous character of the scaling of turbulent velocity fluctuations. The function $\zeta(p)/p$ significantly deviates from Kolmogorov's prediction $\zeta(p)/p = 1/3$, and $\zeta(p)$ depends nonlinearly on p . Assuming that the strongest singularities h_{\min} that can be detected in 1D cuts live in planes, the multifractal model predicts that for large p $\zeta(p)/p \rightarrow h_{\min} + 1/p$. We have indicated this limiting behavior with $h_{\min} = 0.16$ in Fig. 3 [13]. Very recently She and Leveque [14] proposed an inertial range scaling

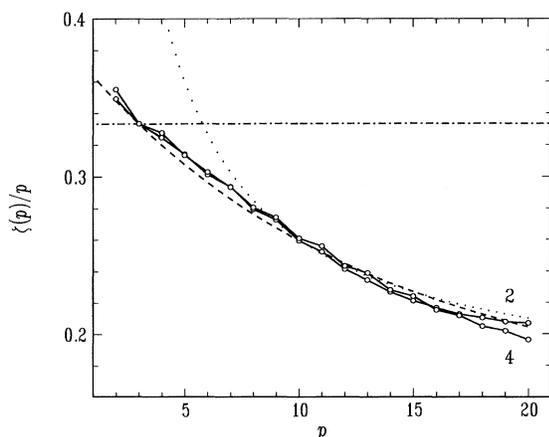


FIG. 3. Full lines: scaling functions $\zeta(p)/p$ in two different turbulent flows. The numbers point to flow conditions in Table I and [11]. Dash-dotted line: Kolmogorov's prediction $\zeta(p)/p = 1/3$; dotted line: $\zeta(p)/p = 1/p + h_{\min}$, with $h_{\min} = 0.16$; dashed line: prediction of She *et al.* [14].

law based on the hypothesis that the most singular fluid structures are filaments and on the assumption of a hierarchy of less singular structures. Their prediction for $\zeta(p)/p$ is also shown in Fig. 3; it fits our data remarkably well.

We believe that the results shown in Fig. 3 are representative of the scaling of turbulent fluctuations at much higher Re_λ . We are not completely certain because the requirements of isotropy and homogeneity are hard to satisfy in laboratory experiments, especially for the large scales. In fact, we have designed our experiments such as to maximize the scaling dynamical range and to satisfy the requirement $\zeta(3) = 1$ as well as possible (it turned out that this was the same as maximizing Re_λ). We estimate that the error bars in Fig. 3 are $\Delta(\zeta(p)/p) = 0.01$ at $p = 10$ and 0.04 at $p = 20$. Apart from experiments mentioned here, we have studied turbulence behind a cylinder and in a boundary layer. In less favorable experiments the determination of $\zeta(p)$ is much more ambiguous and even-order $\zeta(2p)$ and odd-order exponents $\zeta(2p + 1)$ do not fall on a smooth curve.

Scaling arguments do not distinguish between odd and even moments $\langle \Delta u^p \rangle$. The existence of a nonvanishing third-order moment and its scaling behavior follows exactly from the Navier-Stokes equation [1]. There is no such result for the scaling behavior of even-order moments, and only scaling arguments remain. The existence of odd-order moments is due to the skewness of the probability function (PDF) of velocity differences $P(\Delta u)$. This asymmetry between positive and negative velocity differences is nicely illustrated in Fig. 4 that shows $F_r(p) = |\langle \Delta u^p(r) \rangle|^{1/p}$ as a function of p for two values of r that correspond to the bounds of the inertial range. For $p \geq 8$, the $F_r(p)$ at even values of p can, to good approximation, be interpolated between $F_r(p - 1)$ and $F_r(p + 1)$ at neighboring odd values, and vice versa. Therefore, the principal contribution to the high-order moments comes from the *negative velocity increments*.

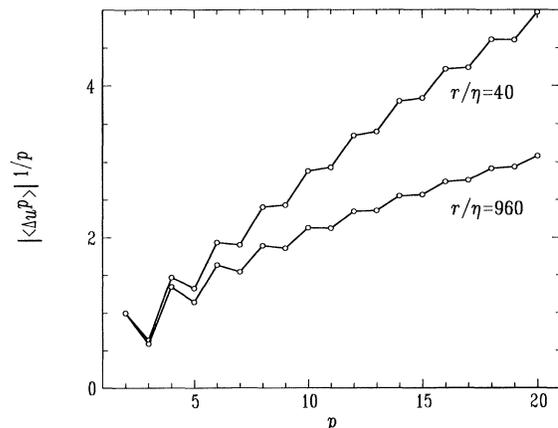


FIG. 4. Structure function of jet turbulence (expt. 4 in Table I) $F_r(p) = |\langle \Delta u^p(r) \rangle|^{1/p}$ as a function of the order p . The normalization is $F_r(2) = 1$.

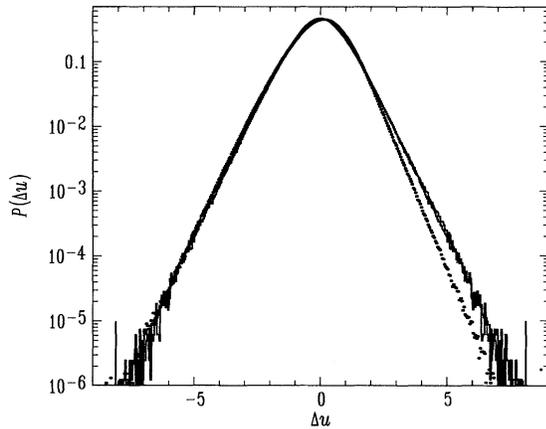


FIG. 5. Probability distribution functions of velocity increments $\Delta u(r)$ at $r/\eta = 90$ in grid turbulence. Full lines transverse $P_t(\Delta u)$ and $P_t(-\Delta u)$, dots: longitudinal $P_l(\Delta u)$. The normalization is $\int P_{l,t}(x)dx = 1$ and $\int x^2 P_{l,t}(x)dx = 1$.

Scaling arguments also do not discriminate between longitudinal Δu_l and transverse Δu_t velocity increments. Transverse velocity differences (where \vec{u} and \vec{r} are perpendicular) have a symmetrical PDF and only even moments exist.

Figure 5 compares $P_l(\Delta u_l)$ and $P_t(\Delta u_t)$ that were measured in grid turbulence and demonstrates that P_l and P_t coincide for $\Delta u < 0$. This coincidence holds in the inertial range (but not outside), and we can expect that also the scaling of high-order moments of transverse and longitudinal velocity fluctuations will be very similar.

Measuring transverse PDF's with two velocity probes is a challenge and requires homogeneity of the flow and careful calibration of the probes. We demonstrate the quality of our result by overlaying $P_t(\Delta u)$ with $P_t(-\Delta u)$.

In conclusion, using a new instrument for the acquisition of high-order structure functions in turbulence, we have for the first time provided evidence for multiscaling of structure functions. Our results also demonstrate that scaling behavior of turbulent velocity fluctuations Δu is a very robust phenomenon; it does not depend on sign or orientation of Δu .

This work is also part of the "Stichting voor Fundamenteel Onderzoek der Materie (FOM)," which is financially supported by the "Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)."

- [1] A.N. Kolmogorov, Dokl. Akad. Nauk. **26**, 115 (1941); see also A.S. Monin and A.M. Yaglom, *Statistical Fluid Mechanics* (MIT Press, Cambridge, 1975), Vol. II.
 [2] F. Anselmet, Y. Gagne, E.J. Hopfinger, and R.A. Anto-

nia, *J. Fluid Mech.* **140**, 63 (1984).

- [3] G. Parisi and U. Frisch, in *Proceedings of the International School in Physics, "E. Fermi," Course LXXXVIII*, edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985), p. 84.
 [4] U. Frisch and M. Vergassola, *Europhys. Lett.* **14**, 439 (1991).
 [5] M.H. Jensen, G. Paladin, and A. Vulpiani, *Phys. Rev. Lett.* **67**, 208 (1991).
 [6] X-Z Wu, L. Kadanoff, A. Libchaber, and M. Sano, *Phys. Rev. Lett.* **64**, 2140 (1990).
 [7] J.A. Herweijer, F.C. van Nijmweegen, K. Kopinga, J.H. Voskamp, and W. van de Water, *Rev. Sci. Instrum.* **65**, 1786 (1994).
 [8] The hot-wire velocity probes have been especially built for this experiment. They consist of a $2.5 \mu\text{m}$ diam Pt-coated tungsten wire that is stretched between two prongs. The sensitive area of the wire is $200 \mu\text{m}$. Usage of a high internal damping material for the prongs (Nimonic) suppressed mechanical resonances. In this way, clean turbulence amplitude spectra that spanned 72 dB could be registered.
 [9] This is the subject of Taylor's frozen turbulence hypothesis. It is valid if the turbulent velocity fluctuations u are relatively small. In our experiments u/U varies approximately from 0.05 to 0.2. Although the applicability of Taylor's hypothesis for measuring high-order moments has not yet been studied carefully, we have not detected a systematic influence of the variation of u/U on our scaling results.
 [10] R. Benzi, S. Ciliberto, R. Tripiccone, C. Baudet, F. Massaioli, and S. Succi, *Phys. Rev. E*, **48**, R29 (1993).
 [11] The numbers in Table I and Fig. 2 point to the following flow realizations. (1) Turbulence 3 m behind a grid with mesh size 0.1 m and 0.36 solidity in a wind tunnel with cross section 0.9 m vertical and 0.7 m horizontal. The probe was located in a region of nonzero mean flow gradient, $h = 0.19$ m from the upper wall. (2) Same as (1) but the vertical size of the wind tunnel test section was reduced to 0.5 m and $h = 0.07$ m. (3) Same as (1) but the mean flow gradient was increased by increasing the solidity of the grid mesh closest to the walls to 0.7. (4) Jet turbulence. The jet flow emanated with 30 m/s from a 12 cm diam jet. Data were taken 2.6 m downstream on the jet center line. (5) Same as (1) but $h = 0.07$ m.
 [12] Such a normalization could be drawing straight lines tangent to $G_p(r)$ in a log-log graph such that the slope $\zeta(3) = 1$; dividing $G_p(r)$ by a lower order structure function [e.g., $G_3(r)$ or $G_2(r)$]; drawing $G_p(r)$ as a function of $|G_3(r)|$ in a log-log plot.
 [13] B.B. Mandelbrot, *Physica (Amsterdam)* **163A**, 306 (1990), points to the possibility of detecting "latent" values of h that are in sets with negative dimensions. This is not corroborated by our results.
 [14] Z-S She and E. Leveque, *Phys. Rev. Lett.* **72**, 336 (1994).