

Constraints on Form Factors for Exclusive Semileptonic Heavy to Light Meson Decays

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We use rigorous QCD dispersion relations to derive model-independent bounds on the $\bar{B} \rightarrow \pi l \bar{\nu}$, $D \rightarrow \pi l \bar{\nu}$, and $D \rightarrow \bar{K} l \bar{\nu}$ form factors. These bounds are particularly restrictive when the value of the observable form factor at one or more kinematic points is assumed. With reasonable assumptions we find $f_B \leq 195$ MeV and that the shape of the form factor becomes severely constrained. These constraints are useful both for model discrimination and for model-insensitive extraction of Cabibbo-Kobayashi-Maskawa mixing parameters.

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(1) *Introduction.*—Charmless B -meson decays are of great interest because the rate depends directly on a fundamental parameter, the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $|V_{ub}|$. Its determination requires knowledge of nonperturbative hadronic matrix elements. Semileptonic decays involve the hadronic matrix element of a partially conserved current, and there is hope that one may calculate them, or at least model them, with some precision. The inclusive charmless decay rate is only measured at $E_e \approx E_{e,\max}$, where the theoretical calculation is highly uncertain. Alternatively, one can measure exclusive rates, e.g., $d\Gamma(\bar{B} \rightarrow \pi e \bar{\nu})$, over the whole kinematic range. One then needs theoretical calculations of the hadronic matrix elements. Similarly, $D \rightarrow \bar{K}$ ($D \rightarrow \pi$) semileptonic decays are interesting because they allow determination of $|V_{cs}|$ ($|V_{cd}|$).

In this Letter we show that one can calculate rather good bounds on the rates for semileptonic exclusive B and D decays to light pseudoscalar mesons. Parametrizing the $B\pi$ matrix element of the flavor-changing vector current $V_\mu = \bar{u}\gamma_\mu b$ by

$$\langle \pi(p') | V_\mu | \bar{B}(p) \rangle = f_+(q^2)(p + p')_\mu + f_-(q^2)(p - p')_\mu, \quad (1.1)$$

we obtain inequalities of form

$$F_-(q^2) \leq |f_+(q^2)| \leq F_+(q^2). \quad (1.2)$$

Below we describe the calculation of the functions F_\pm . The bounds are model independent. They involve a few physical parameters: masses, decay constants, and the $B^*-B\pi$ coupling $g_{B^*B\pi}$ that must be determined independently. In addition, for a strong bound one needs the value of the form factor for at least one kinematic point, but this may not require additional parameters.

The method we will employ is not new [1]. It was used to obtain bounds on form factors for semileptonic K -meson decays [2]. The method has also been applied to the decay $B \rightarrow \bar{D} e \nu$ [3], but here there is an important difference [4,5]. While there are no poles below the onset of vacuum $\rightarrow \bar{K}\pi$, there are several resonances with masses smaller than $m_B + m_D$, namely, the oniumlike B_c 's. As pointed out in Ref. [5], the case $\bar{B} \rightarrow \pi e \bar{\nu}$ is intermediate

between these: There is exactly one resonance below the onset of the $\bar{B}\pi$ continuum, the \bar{B}^* . This is phenomenologically true. It is also guaranteed in the heavy quark limit for m_π small and fixed, since the B^*-B mass splitting is $\mathcal{O}(1/m_B)$. In the case $B \rightarrow \bar{D}$ the multitude of resonances below $m_B + m_D$ renders the method quite weak, even though heavy quark symmetries fix the values of the form factors at one kinematic point. For $B \rightarrow \pi$ the situation is improved because there is only one such resonance.

(2) *Method.*—The derivation of the bounds is well known. We present a short version here both to establish notation and to underline where we may deviate from the standard case. Consider the two-point function

$$i \int d^4x e^{iqx} \langle TV_\mu(x) V_\nu^\dagger(0) \rangle = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi_T(q^2) + g_{\mu\nu} \Pi_L(q^2). \quad (2.1)$$

In QCD the structure functions satisfy a once-subtracted dispersion relation

$$\chi_{T,L}(Q^2) = \frac{\partial \Pi_{T,L}}{\partial q^2} \Big|_{q^2=-Q^2} = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im} \Pi_{T,L}(t)}{(t + Q^2)^2}. \quad (2.2)$$

The absorptive parts $\text{Im} \Pi_{T,L}(q^2)$ are obtained by inserting real states between the two currents on the right-hand side of Eq. (2.1). A judicious choice of μ and ν makes this a sum of positive definite terms, so one can obtain strict inequalities by concentrating on the term with intermediate states of $\bar{B}\pi$ pairs. For Q^2 far from the resonance region the two-point function can be computed reliably from perturbative QCD. In particular, for large b quark mass, $Q^2 = 0$ is far from resonances. One resulting inequality of this method is

$$\int_{t_+}^\infty dt k(t) |f_+(t)|^2 \leq 1, \quad (2.3)$$

where, neglecting the light quark mass,

$$k(t) = \frac{1}{3}(m_b/t)^2 [(1 - t_+/t)(1 - t_-/t)]^{3/2}, \quad (2.4)$$

and $t_\pm = (m_B \pm m_\pi)^2$.

Using knowledge of the analytic structure of the form factor plus the bound Eq. (2.3) one can derive bounds [1,2] on the form factor in the physical region of semileptonic decay, $0 \leq t \leq t_-$. To this end we map the complex t plane onto the unit disk $|z| \leq 1$ by the transformation

$$\sqrt{\frac{t_+ - t}{t_+ - t_-}} = \frac{1+z}{1-z}. \quad (2.5)$$

The two branches of the root for $t_+ \leq t$ are mapped into the unit circle $z = e^{i\theta}$, while the regions $t \leq t_-$ and $t_- \leq t < t_+$ are mapped into the segments of the real axis $0 \leq z < 1$ and $-1 < z \leq 0$, respectively. In terms of this new variable the inequality (2.3) is

$$\frac{1}{2} \int_0^{2\pi} d\theta w(\theta) |f_+|^2 \leq 1, \quad (2.6)$$

where $w(\theta) = k(t(\theta))dt/d\theta$. Next we construct a function $\phi(z)$ analytic in $|z| < 1$ such that $|\phi(e^{i\theta})|^2 = w(\theta)$:

$$\phi(z) = \frac{2^{5/2} m_b (t_+ - t_-)^{-1/2} (1+z)^2}{\sqrt{3} (1-z)^{9/2}} \left(\beta_+ + \frac{1+z}{1-z} \right)^5, \quad (2.7)$$

where $\beta_+ = \sqrt{t_+/(t_+ - t_-)}$.

With Ref. [2], let us define an inner product on the space of complex functions of a real variable θ , with $0 \leq \theta < 2\pi$, by

$$(f, g) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta f^*(\theta) g(\theta). \quad (2.8)$$

Next let

$$\begin{aligned} f_0(\theta) &= \phi(e^{i\theta}) f_+(e^{i\theta}), \\ f_i(\theta) &= \frac{1}{1 - \bar{z}_i e^{i\theta}}, \end{aligned} \quad (2.9)$$

where z_i are arbitrary complex numbers with $|z_i| < 1$. With this, we have

$$I \equiv (f_0, f_0) = \frac{1}{2\pi} \int_0^{2\pi} d\theta w(\theta) |f_+|^2 \leq \frac{1}{\pi}. \quad (2.10)$$

Using Cauchy's theorem we can evaluate the other inner products. We must bear in mind that the form factor f_+ has a pole at $t = t_* \equiv m_{B^*}^2$, corresponding to $z_* = z(t = m_{B^*}^2)$ inside the unit circle. For example,

$$(f_1, f_0) = \phi(z_1) f_+(z_1) + \frac{\text{Res}(\phi f_+)|_{z_*}}{z_* - z_1}. \quad (2.11)$$

From the positivity of the inner product we have that the matrix (f_i, f_j) has positive determinant. Inequalities (1.2) follow; it is straightforward to display analytic expressions for the bounding functions F_{\pm} . We can further improve our bounds by including the vector meson contribution to the absorptive part of the structure functions, and by generalizing the calculation to nonzero Q^2 .

As a side benefit we find that $f_{B^*} < 1/4\pi\sqrt{3/2} m_{B^*}^3/m_b$. This is consistent with the heavy quark symmetry relation $f_{B^*} = (m_B m_D)^{1/2} f_D$ and the bound $f_D < m_D/4\pi$ from an analogous dispersion relation [6].

(3) *Analysis and discussion.*— $B \rightarrow \pi$: The bounds on f_+ require explicit knowledge of the residue $F_* = f_{B^*} g_{B^* B \pi}$. Heavy quark symmetries imply $g_{B^* B \pi}/m_B = g_{D^* D \pi}/m_D$ and $f_{B^*} = m_B f_B$, at leading order. An experimental upper bound on the D^* width [7], together with measurements of the D^* decay fractions [8], gives (using 90% confidence values) [9] $0.06 \leq g^2 \leq 0.5$, where $g = f_{\pi} g_{D^* D \pi}/m_D$ to leading order in heavy meson chiral perturbation theory [10]. Monte Carlo simulations of quenched lattice QCD give $f_B(\text{MeV})$ in the range [11] 150–290 with about 20% errors, and an unquenched calculation gives [12] 200 ± 48 . Clearly F_* is poorly known. In what follows we shall take $F_* = 33 \text{ GeV}^2$, corresponding to $g^2 = 0.5$ and $f_B = 220 \text{ MeV}$. Our bounds are stronger for larger F_* , so the value we have chosen is not conservative, but rather intended to illustrate the potential of the method. We also take $Q^2 = -16 \text{ GeV}^2$, which is chosen to be closer to the resonance region without violating our perturbative QCD assumption. This typically narrows the band between the upper and lower bounds by (10–15)%. The results of Ref. [2] may be used to gauge the reliability of this choice of Q^2 . In addition, we include the contribution of the B^* to our dispersion relation, but the resulting improvement is typically only a few percent.

Figure 1(a) shows as solid lines the upper and lower bounds from the 2×2 determinant. The abscissa in all B -meson plots is presented in units of t/M_B^2 . For reference we have plotted as dashed lines a “pure pole” form factor $f_{\text{pole}}(t) = F_*/(m_{B^*}^2 - t)$. Although not very stringent, this bound uses the minimal set of assumptions and could be used to put a rigorous lower bound on $|V_{ub}|$ from a measurement of the width of $\bar{B} \rightarrow \pi e \bar{\nu}$.

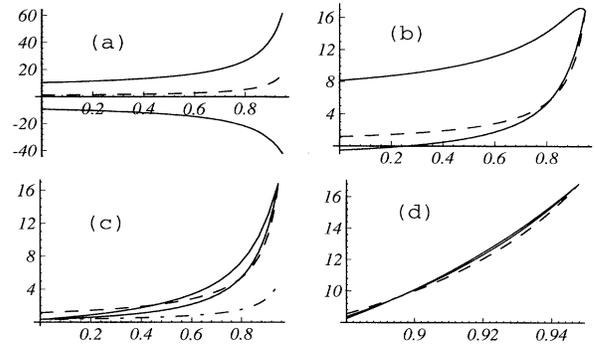


FIG. 1. Upper and lower bounds (solid lines) on $f_+(t)$ for $B \rightarrow \pi$, plotted against t/M_B^2 . The pure pole form factor $f_{\text{pole}}(t)$ is plotted as dashed lines, while the WSB model is in dot-dashed lines. The bound is from the (a) 2×2 , (b) 3×3 , and (c), (d) 4×4 determinants. In (b) we assume $f_+(t_-) \approx f_{\text{pole}}(t_-)$. In (c) we use $f_+(0)$ from the WSB model and $f_+(t_-)$ from B^* pole dominance. In (d) we use as inputs $f_+(t_-)$ and $f_+(t_- - 2m_B m_\pi)$ from the pole dominance assumption of heavy meson chiral perturbation theory. At the scale of the figure the bounds are indistinguishable.

Bounds using the value of f_+ at one or more points are significantly more restrictive. The proximity of the B^* pole to the region of maximum momentum transfer suggests $f_+(t_-) = f_{\text{pole}}(t_-)$ to good approximation. We make this assumption in Fig. 1(b), which requires a 3×3 determinant. The dashed line shows the simple pole curve, which, remarkably, falls outside the region allowed by our bounds for values of momentum transfer close to t_- . We find one must decrease the value of F_* to 23 GeV^2 before the pole term lies entirely within the allowed region.

There are several models for f_+ in the literature. They are intended to give a numerical approximation to the actual form factor in the physical region for $\bar{B} \rightarrow \pi e \bar{\nu}$. One can test whether a particular model is consistent with QCD by using an arbitrary number of points $f_+(t_i)$ in our bounds. We will content ourselves with bounds that use the value of f_+ at two points. This requires a computation with a 4×4 determinant. We take $f_+(t_-) = f_{\text{pole}}(t_-)$ as above, and fix a second point $f_+(t_i)$ from the model under scrutiny.

The model of Wirbel, Stech, and Bauer (WSB) has $f_+(0) = 0.33$, and assumes a single B^* -pole shape [13]. Presumably it is not intended to describe the form factor accurately as $t \rightarrow t_-$ [14]. Figure 1(c) shows the bounds obtained using $f_+(0)$ from this model as a solid line, the pure pole $f_{\text{pole}}(t)$ as a dashed line, and the WSB model prediction as a dot-dashed line. For the given value of F_* , WSB falls outside of our bounds over the entire physical range. For $F_* < 23 \text{ GeV}^2$, the WSB curve lies within the bounds over a range from $t = 0$ to some t_{crit} , where t_{crit} increases as F_* decreases. A revised version of the model of Scora and Isgur [15] gives a somewhat smaller form factor for $B \rightarrow \pi$, leading to a smaller value of t_{crit} .

The validity of chiral perturbation theory for heavy mesons hinges on single pole dominance of f_+ at and near $t = t_-$. Figure 1(d) shows the bounds using input normalizations $f_+(t_-) = f_{\text{pole}}(t_-)$ and $f_+(t_- - 2m_B m_\pi) = f_{\text{pole}}(t_- - 2m_B m_\pi)$. This simply assumes heavy meson chiral perturbation theory is valid at both $E_\pi = m_\pi$ and $2m_\pi$. The pure pole is again shown as a dashed line. Either the effects of higher resonances are non-negligible, or the value of F_* is inconsistent with chiral perturbation theory. Insisting on the validity of heavy meson chiral perturbation theory in this range implies an upper bound, $F_* \leq 10 \text{ GeV}^2$. Substituting the lower bound in Ref. [9] for g^2 then gives $f_B \leq 195 \text{ MeV}$.

$D \rightarrow K$, $D \rightarrow \pi$: For $D^+ \rightarrow \pi^0$, $D^0 \rightarrow \pi^-$, and $D_s^+ \rightarrow \pi^0$ we have $m_{D^*} > m_D + m_\pi$ so we need no *a priori* knowledge of the residue F_* of the vector meson pole. However, useful bounds are obtained only if one has additional information about the form factors. $D \rightarrow K$ form factors have one pole below threshold, and must be treated like those of $B \rightarrow \pi$.

Assuming the value of the form factor for $D^0 \rightarrow \pi^-$ is dominated by the D^{*+} pole at $t = t_- \equiv (m_D - m_\pi)^2$ gives the bound in Fig. 2. We have taken $Q^2 = 0$

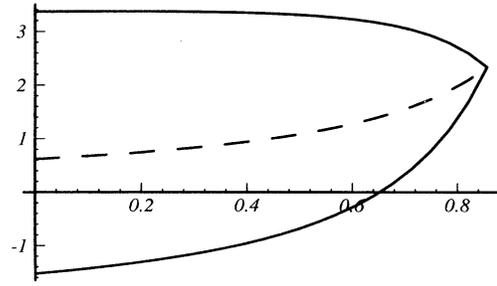


FIG. 2. Upper and lower bounds on $f_+(t)$ for $D \rightarrow \pi$, assuming $f_+(t_-) = F_*/(m_{D^*}^2 - t_-)$. The curve $F_*/(m_{D^*}^2 - t)$ is shown as a dashed line. The abscissa is given in units of t/M_D^2 .

and $F_* = 2.5 \text{ GeV}^2$, and plotted the pure pole in dashes. A more restrictive bound follows from using two normalization points, as in the B -meson analysis. However, the perturbative QCD calculation is less reliable than in the B -meson case.

The experimental measurements of f_D , [16] are one to two standard deviations from the bound of Ref. [6]. How this bound eventually fares will shed light on the minimal value of Q^2 consistent with reliable limits on $f_+^{D \rightarrow \pi}(t)$.

In summary, the analytic structure of form factors for heavy to light semileptonic meson decays makes them well suited to analysis by simple dispersion relations. The validity of this analysis depends on the use of perturbative QCD calculations at a distance $M_{B^*, D^*}^2 + Q^2$ from the resonance region.

For $\bar{B} \rightarrow \pi l \bar{\nu}$ decays, only the value of the product of the decay constant f_{B^*} and the coupling $g_{B^* B \pi}$, $F_* = f_{B^*} g_{B^* B \pi}$, is necessary for model-independent bounds on the experimentally accessible pion form factor $f_+(q^2)$. For $D \rightarrow \pi l \bar{\nu}$ and $D \rightarrow \bar{K} l \bar{\nu}$ decays, even this input is unnecessary. Together with experimental data, these form factor bounds yield model-independent lower bounds on the Cabibbo-Kobayashi-Maskawa parameters $|V_{ub}|$, $|V_{cs}|$, and $|V_{cd}|$.

Much more restrictive form factor bounds result if the value of $f_+(q^2)$ is known at a single kinematic point. This normalization may come from experiment, lattice calculations, or phenomenological and QCD-inspired models. These form factor bounds allow the experimental extraction of both upper and lower bounds on CKM angles, and place significant restrictions on models. For example, using a one-point normalization, we show that heavy meson chiral perturbation theory with minimal particle content is inconsistent for values of $F_* > 23 \text{ GeV}^2$.

Using the normalization of $f_+(q^2)$ at two kinematic points yields even more restrictive form factor bounds. Typically the shape of the form factor between the normalization points is very severely constrained. This can be used to interpolate between models in disparate regions of phase space, or to restrict the parameter space of a given model. In the case of heavy meson chiral

perturbation theory, we find consistency only if $F_* < 10 \text{ GeV}^2$. This translates into the prediction $f_B < 195 \text{ MeV}$. Similar analyses may be applied to other models. We hope to present the consequences of our bounds more thoroughly in a future work.

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