PHYSICAL REVIEW LETTERS

VOLUME 74 | 5 JUNE 1995 | NUMBER 23

Local Manley-Rowe Relations for Noneikonal Wave Fields

A. J. Brizard and A. N. Kaufman
Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720
(Received 23 February 1995)

The wave-action conservation laws for three-wave interactions are of fundamental importance. These laws, known as Manley-Rowe relations, are usually obtained with great difficulty from nonlinear evolution equations. Here, we derive them directly from a Noether symmetry of the appropriate Lagrangian. As an example, the Lagrangian formulation for stimulated Raman scattering is presented, and local Manley-Rowe relations, valid even for noneikonal wave fields, are derived by the Noether method.

PACS numbers: 03.40.Kf, 52.35.Mw, 52.35.Nx

Manley-Rowe relations are wave-action conservation laws relating to the nonlinear process of three-wave interaction [1]. Most derivations [1,2] have involved a large amount of algebra, and have been motivated by the intuitive confidence that these relations are always valid. However, a concise law should have a concise derivation; in particular, a conservation law should proceed from a Noether symmetry of an underlying Lagrangian [3].

In this paper, we identify the symmetry operation as a phase shift of a *complex representation* of the wave fields [4]. The Lagrangian density for this representation is obtained by performing a Whitham average [5] of the real-field Lagrangian, based on the existence of two separate time scales: a short time scale represented by the wave frequencies, and a long time scale associated with the evolution of the wave action and of the background medium. This approach allows us to generalize the applicability of the Manley-Rowe relations to noneikonal conditions, i.e., conditions for which the eikonal approximation [6] is invalid, such as when wavelengths become comparable to the background gradient scale length.

We allow the background medium to have arbitrary spatial variation, so that the concept of local wave vector need not be valid. Our aim is to obtain explicit local conservation laws, in terms of gradients of the wave fields, rather than in terms of wave vectors. We also allow for a slow temporal variation of the background medium, as expected physically from the ponderomotive effects of the waves. The slowness is required for the utility of the complex representation for the wave fields. In addition, we do not require the waves to satisfy local dispersion relations, thus allowing for strong turbulence.

Our basic assumption is that the frequency spectrum of the fields $\Psi^i(\mathbf{x},t)$ ($i = 1,\ldots,N$; N is the number of field components) allows a clear separation between a quasistatic part $\Psi_0^i(\mathbf{x},t)$ and a finite-frequency part, $\psi^i(\mathbf{x},t) + \psi^{i*}(\mathbf{x},t)$, expressed as twice the real part of its analytic signal. (The latter is defined as the Fourier integral over positive frequencies, with a low-frequency cutoff.) Thus we have, for each field component,

$$\Psi^i = \Psi_0^i + \psi^i + \psi^{i*}. \tag{1}$$

Since the Manley-Rowe relations refer to wave fields in three frequency bands, centered at $(\omega_1, \omega_2, \omega_3)$, satisfying an approximate resonance condition:

$$\omega_1 + \omega_2 \simeq \omega_3, \tag{2}$$

we further assume that the analytic signal ψ^i consists of three parts:

$$\psi^i = \sum_{\alpha=1}^{3} \psi_\alpha^i, \tag{3}$$

where ψ_1^i represents the Fourier integral over the band at ω_1, and so on. Our aim is to derive local Manley-Rowe relations of the form:

$$\partial_t[J_1(\mathbf{x},t) - J_2(\mathbf{x},t)] = -\nabla\cdot[\mathbf{\Gamma}_1(\mathbf{x},t) - \mathbf{\Gamma}_2(\mathbf{x},t)], \tag{4a}$$

$$\partial_t[J_3(\mathbf{x},t) + J_1(\mathbf{x},t)] = -\nabla\cdot[\mathbf{\Gamma}_3(\mathbf{x},t) + \mathbf{\Gamma}_1(\mathbf{x},t)], \tag{4b}$$

$$\partial_t[J_3(\mathbf{x},t) + J_2(\mathbf{x},t)] = -\nabla\cdot[\mathbf{\Gamma}_3(\mathbf{x},t) + \mathbf{\Gamma}_2(\mathbf{x},t)], \quad (4c)$$

with explicit local expressions for the three wave-action densities $J_\alpha(\mathbf{x},t)$ and action flux densities $\mathbf{\Gamma}_\alpha(\mathbf{x},t)$.

We begin with a Lagrangian density

$$\mathcal{L}(\mathbf{x},t) \equiv \mathcal{L}(\Psi^i(\mathbf{x},t), \partial_t\Psi^i(\mathbf{x},t), \nabla\Psi^i(\mathbf{x},t); \mathbf{x},t), \quad (5)$$

and first consider the case that $\mathcal{L}$ is *multilinear* in the fields. Consider, for example, a trilinear term:

$$\beta_{ijk}\Psi^i\Psi^j\Psi^k, \quad (6)$$

where β is a differential operator, possibly with explicit $(\mathbf{x},t)$ dependence, but *quasistatic* in time. For each component field we substitute (1) and (3) into (6), and collect the 343 ($= 7^3$) terms. We then apply Whitham averaging [5], defined as an integration over the short time scales represented by the wave frequencies, and find that most of the terms vanish. Those that survive (a total of 31) have only a slow time-scale dependence, and fall into three groups. The first group consists only of the *background* term, $\beta_{ijk}\Psi_0^i\Psi_0^j\Psi_0^k$, the second group consists of *bilinear* wave-field terms belonging to the *same* frequency band:

$$\beta_{ijk}\sum_{\alpha=1}^{3}(\Psi_0^i\psi_\alpha^j\psi_\alpha^{k*} + \Psi_0^j\psi_\alpha^k\psi_\alpha^{i*} + \Psi_0^k\psi_\alpha^i\psi_\alpha^{j*} + \text{c.c.}), \quad (7)$$

and the third group consists of *trilinear* wave-field terms belonging to three *separate* frequency bands satisfying the resonance condition (2):

$$\beta_{ijk}\sum_{\alpha=1}^{3}\sum_{\Delta}(\psi_\alpha^i\psi_\beta^j\psi_\gamma^{k*} + \text{c.c.}), \quad (8)$$

where the summation $\sum_\Delta$ is over all possible combinations $\alpha + \beta = \gamma$, such as $1 + 2 = 3$ and $2 - 3 = -1$, with $\psi_{-\alpha} \equiv \psi_\alpha^*$. For more general Lagrangian densities, after Ψ^i is written as in (1) and (3), the Lagrangian density $\mathcal{L}$ is Taylor expanded in powers of the wave fields and, upon Whitham averaging, yields the Whitham-averaged Lagrangian density $\overline{\mathcal{L}} \equiv \overline{\mathcal{L}}_0 + \overline{\mathcal{L}}_{II} + \overline{\mathcal{L}}_{III} + \cdots$, where the linear term $\overline{\mathcal{L}}_I$ is absent.

To derive the Manley-Rowe relation (4a), we note that the Whitham-averaged Lagrangian $\overline{\mathcal{L}}$ is *invariant* under the phase shifts:

$$\psi_1^j \to \psi_1^j e^{i\phi}, \qquad \psi_2^j \to \psi_2^j e^{-i\phi}, \qquad \psi_3^j \to \psi_3^j, \quad (9)$$

where ϕ is a real constant; note that (7) and (8) are both invariant under these phase shifts. For infinitesimal ϕ, the variation in $\overline{\mathcal{L}}$ is thus:

$$0 = \delta\overline{\mathcal{L}} \equiv \sum_{\alpha,j}\left(\frac{\partial\overline{\mathcal{L}}}{\partial\psi_\alpha^j}\,\delta\psi_\alpha^j + \frac{\partial\overline{\mathcal{L}}}{\partial\psi_{\alpha,t}^j}\,\delta\psi_{\alpha,t}^j + \frac{\partial\overline{\mathcal{L}}}{\partial\nabla\psi_\alpha^j}\cdot\delta\nabla\psi_\alpha^j + \text{c.c.}\right). \quad (10)$$

Using the Euler-Lagrange equations,

$$\frac{\partial\overline{\mathcal{L}}}{\partial\psi_\alpha^j} = \frac{\partial}{\partial t}\left(\frac{\partial\overline{\mathcal{L}}}{\partial\psi_{\alpha,t}^j}\right) + \nabla\cdot\left(\frac{\partial\overline{\mathcal{L}}}{\partial\nabla\psi_\alpha^j}\right), \quad (11)$$

and the variations $\delta\psi_1^j = i\phi\psi_1^j, \delta\psi_2^j = -i\phi\psi_2^j$, and $\delta\psi_3^j = 0$, we obtain the desired result (4a) from (10), with the explicit expressions:

$$J_\alpha(\mathbf{x},t) \equiv 2\sum_{j=1}^{N}\text{Im}\left(\psi_\alpha^j\frac{\partial\overline{\mathcal{L}}}{\partial\psi_{\alpha,t}^j}\right),$$
$$\mathbf{\Gamma}_\alpha(\mathbf{x},t) \equiv 2\sum_{j=1}^{N}\text{Im}\left(\psi_\alpha^j\frac{\partial\overline{\mathcal{L}}}{\partial\nabla\psi_\alpha^j}\right), \quad (12)$$

where Im denotes the imaginary part. Here, the time dependence of the wave-action density $J_\alpha(\mathbf{x},t)$ and the wave-action flux $\mathbf{\Gamma}_\alpha(\mathbf{x},t)$ is over *long* time scales, and is independent of the *short* time scales represented by the wave frequencies. Notice that the complex wave-field representation is essential for obtaining nonvanishing expressions for the wave-action density and wave-action flux. The factor 2 appearing in (12) ensures that the wave-action densities and fluxes as defined here agree with those obtained in the eikonal limit [see (21) below]. The other two Manley-Rowe relations (4b) and (4c) can be obtained in a similar fashion.

To illustrate the Manley-Rowe relations (4a)–(4c), we consider an isentropic electron fluid model (with *one* spatial dimension and time) to represent the stimulated Raman scattering (SRS) process, involving the interaction of two transverse electromagnetic waves and a longitudinal Langmuir wave [7]. For a Lagrangian formulation of SRS within this one-dimensional model, we use the four-component field $\Psi^i \equiv (n, \chi, \Phi, A)$, where $n(x,t)$ is the electron-fluid density, $\chi(x,t)$ is the electron-fluid velocity potential, $\Phi(x,t)$ is the electrostatic potential, and $\mathbf{A}(x,t) \equiv A(x,t)\hat{\mathbf{y}}$ is the magnetic vector potential (corresponding to the gauge choice $\nabla\cdot\mathbf{A} = 0$). Here, the electron-fluid velocity $\mathbf{v}$ is expressed in terms of χ and A as $\mathbf{v} \equiv u\hat{\mathbf{x}} + \hat{\mathbf{y}}eA/mc$, with $u = \partial_x\chi$, the electric and magnetic fields are $\mathbf{E} = -\hat{\mathbf{x}}\partial_x\Phi - \hat{\mathbf{y}}\partial_t A/c, \mathbf{B} = \hat{\mathbf{z}}\partial_x A$, and $(-e, m)$ are the charge and mass of an electron. In the case of an isentropic fluid, the specific energy, denoted ε, depends only on the fluid density.

In SRS, the Langmuir wave and the two electromagnetic waves satisfy the resonance condition (2). The Langmuir wave, identified as wave 1, is described by the wave field $\psi_1^i = (n_1, \chi_1, \Phi_1, 0)$. The electromagnetic waves, identified as waves 2 and 3, are described by the wave field $\psi_s^i = (0,0,0,A_s)$ (for $s = 2$ or 3). The Lagrangian density is [8]

$$\mathcal{L} = \frac{1}{8\pi}\left[\frac{1}{c^2}\left(\frac{\partial A}{\partial t}\right)^2 + \left(\frac{\partial\Phi}{\partial x}\right)^2 - \left(\frac{\partial A}{\partial x}\right)^2\right] - \frac{e^2 n}{2mc^2}A^2 - mn\frac{\partial\chi}{\partial t} - \frac{mn}{2}\left(\frac{\partial\chi}{\partial x}\right)^2 + e(n - N_i)\Phi - mn\varepsilon(n), \quad (13)$$

where $N_i(x,t)$ is the prescribed ion-fluid density. The nonlinear evolution equation for each component field is expressed as an Euler-Lagrange equation obtained from the variational principle $\delta(\int dx\, dt\, \mathcal{L}) = 0$.

Writing Ψ^i as in (1) and (3), we Taylor-expand (13) in powers of the wave fields, and after performing a Whitham average, we obtain $\overline{\mathcal{L}} \equiv \overline{\mathcal{L}}_0 + \overline{\mathcal{L}}_{II} + \overline{\mathcal{L}}_{III} + \cdots$; the bilinear term is

$$\overline{\mathcal{L}}_{II} = \sum_{s=2}^{3} \frac{1}{4\pi}\left(\frac{1}{c^2}\left|\frac{\partial A_s}{\partial t}\right|^2 - \left|\frac{\partial A_s}{\partial x}\right|^2 - \frac{\omega_p^2}{c^2}|A_s|^2\right) + \frac{1}{4\pi}\left|\frac{\partial \Phi_1}{\partial x}\right|^2 + 2e\,\mathrm{Re}(n_1\Phi_1^*)$$
$$- 2m\,\mathrm{Re}\left[n_1\left(\frac{\partial \chi_1^*}{\partial t} + u_0\frac{\partial \chi_1^*}{\partial x}\right)\right] - mn_0\left|\frac{\partial \chi_1}{\partial x}\right|^2 - \frac{mv_t^2}{n_0}|n_1|^2, \tag{14}$$

where $v_t \equiv \{n_0 d^2[n_0\varepsilon(n_0)]/dn_0^2\}^{1/2}$ is an electron thermal speed and $\omega_p \equiv (4\pi n_0 e^2/m)^{1/2}$ is the electron plasma frequency, while the trilinear term is

$$\overline{\mathcal{L}}_{III} = -\frac{e^2}{mc^2}\mathrm{Re}(n_1 A_2 A_3^*). \tag{15}$$

In Eq. (14), (n_0, u_0) are functions of x and t; the ion-fluid density N does not appear in (14), as a result of Whitham averaging. Using the result (12), we then find the Langmuir wave-action density

$$J_1(x,t) = 2m\,\mathrm{Im}(n_1\chi_1^*), \tag{16}$$

and Langmuir wave-action flux

$$\Gamma_1(x,t) = u_0 J_1 + 2mn_0\mathrm{Im}\left(\chi_1^*\frac{\partial\chi_1}{\partial x}\right) - \frac{1}{2\pi}\mathrm{Im}\left(\Phi_1^*\frac{\partial\Phi_1}{\partial x}\right), \tag{17}$$

while the transverse-wave action density and flux ($s = 2$ or 3) are

$$J_s(x,t) = \frac{1}{2\pi c^2}\mathrm{Im}\left(A_s\frac{\partial A_s^*}{\partial t}\right), \tag{18}$$

$$\Gamma_s(x,t) = \frac{1}{2\pi}\mathrm{Im}\left(A_s^*\frac{\partial A_s}{\partial x}\right). \tag{19}$$

Note that because the trilinear term (15) does not contain any time derivatives or spatial gradients of the wave field, it does not contribute to the expressions (16)–(19) for the wave-action density and wave-action flux.

To verify that the Manley-Rowe relations (4a)–(4c) are satisfied for our one-dimensional SRS model, we explicitly evaluate $\partial_t J_1(x,t)$, $\partial_t J_2(x,t)$, and $\partial_t J_3(x,t)$, using the nonlinear evolution equations for each wave (which include terms involving gradients of the background fields). Without giving details, we find the *exact* equations:

$$\partial_t J_1(x,t) = -\partial_x\Gamma_1(x,t) - \eta(x,t), \tag{20a}$$

$$\partial_t J_2(x,t) = -\partial_x\Gamma_2(x,t) - \eta(x,t), \tag{20b}$$

$$\partial_t J_3(x,t) = -\partial_x\Gamma_3(x,t) + \eta(x,t), \tag{20c}$$

where $\eta(x,t) \equiv (e^2/mc^2)\mathrm{Im}[n_1 A_2 A_3^*](x,t)$ is the nonlinear coupling term. It is then easy to verify from these equations that the Manley-Rowe relations (4a)–(4c) are indeed satisfied.

We now check that our expressions (16)–(19) for the wave-action density and wave-action flux of the Langmuir wave and transverse waves have the correct eikonal limit [6]. In this limit, the complex wave fields are represented as $\psi_\alpha^j(\mathbf{x},t) = \tilde{\psi}_\alpha^j(\mathbf{x},t)\exp i\theta_\alpha(\mathbf{x},t)$, where derivatives of the eikonal phase θ_α define the local wave frequency $\omega_\alpha(\mathbf{x},t) \equiv -\partial_t\theta_\alpha$ and the local wave vector $\mathbf{k}_\alpha(\mathbf{x},t) \equiv \nabla\theta_\alpha$; here, the wave amplitude $\tilde{\psi}_\alpha^j$, wave frequency ω_α, and wave vector $\mathbf{k}_\alpha$ are slowly varying functions of $\mathbf{x}$ and t. Within the *eikonal* representation, the wave-action density and wave-action flux are now defined [3] from the eikonal-phase-averaged Lagrangian density $\langle\mathcal{L}\rangle(\omega_\alpha, \mathbf{k}_\alpha, \tilde{\psi}_\alpha; \mathbf{x}, t)$ as

$$J_\alpha(\mathbf{x},t) \equiv \frac{\partial\langle\mathcal{L}\rangle}{\partial\omega_\alpha} \quad\text{and}\quad \boldsymbol{\Gamma}_\alpha(\mathbf{x},t) \equiv -\frac{\partial\langle\mathcal{L}\rangle}{\partial\mathbf{k}_\alpha}. \tag{21}$$

Using the eikonal representation for the one-dimensional SRS wave fields, (14) becomes

$$\langle\mathcal{L}_{II}\rangle = \sum_s(\omega_s^2 - k_s^2c^2 - \omega_p^2)\frac{|\tilde{A}_s|^2}{4\pi c^2} + \frac{k_1^2}{4\pi}|\tilde{\Phi}_1|^2 + 2e\,\mathrm{Re}(\tilde{n}_1\tilde{\Phi}_1^*) + 2m\overline{\omega}_1\mathrm{Im}(\tilde{n}_1\tilde{\chi}_1^*) - mn_0k_1^2|\tilde{\chi}_1|^2 - (mv_t^2/n_0)|\tilde{n}_1|^2, \tag{22}$$

where $\overline{\omega}_1 \equiv \omega_1 - k_1u_0$ is the Langmuir-wave frequency in the local rest frame, while (15) becomes $\langle\mathcal{L}_{III}\rangle = -(e^2/mc^2)\mathrm{Re}(\tilde{n}_1\tilde{A}_2\tilde{A}_3^*)$. Using (21) and (22), the Langmuir-wave action density (16) and action flux (17) become

$$J_1 = 2mn_0|\tilde{u}_1|^2/\overline{\omega}_1, \tag{23}$$

and

$$\Gamma_1 = (u_0 + k_1v_t^2/\overline{\omega}_1)J_1, \tag{24}$$

respectively, where $\tilde{n}_1 = n_0k_1\tilde{u}_1/\overline{\omega}_1$ and $\tilde{\Phi}_1 = -4\pi\tilde{n}_1e/k_1^2$ have been eliminated in favor of $\tilde{u}_1 \equiv ik_1\tilde{\chi}_1$ through the Langmuir-wave dispersion relation: $\omega_1 = k_1u_0 \pm (k_1^2v_t^2 + \omega_p^2)^{1/2}$. Substituting the group velocity $v_{g1} \equiv \partial\omega_1/\partial k_1 = u_0 + k_1v_t^2/\overline{\omega}_1$ into (24), we obtain the standard relation: $\Gamma_1 = v_{g1}J_1$. The standard relation $J_1 = E_1/\omega_1$ for the Langmuir-wave action density requires the eikonal-phase-averaged bilinear energy density E_1 for the Langmuir wave. Here, the bilinear eikonal-phase-averaged Lagrangian density (22) is written as $\langle\mathcal{L}_{II}\rangle \equiv \sum_\alpha L_\alpha$ and the bilinear eikonal-phase-averaged energy is defined [3] as $E_\alpha \equiv \omega_\alpha\partial L_\alpha/\partial\omega_\alpha - L_\alpha$. For the Langmuir wave, we find $E_1 = 2mn_0|\tilde{u}_1|^2(\omega_1/\overline{\omega}_1)$, and, hence, the Langmuir-wave action density (23) can also be expressed as $J_1 = E_1/\omega_1$. We note that, although E_1 and ω_1 are not Galilean invariant, (23) shows that the Langmuir-wave action density is, in fact, Galilean invariant.

Next, the transverse-wave action density (d18) and flux (19) become in the eikonal limit: $J_s = \omega_s|\tilde{A}_s|^2/2\pi c^2$ and $\Gamma_s = k_s|\tilde{A}_s|^2/2\pi$, respectively. Using the transverse-wave dipersion relation $\omega_s = \pm(k_s^2c^2 + \omega_p^2)^{1/2}$, we find $v_{gs} \equiv k_sc^2/\omega_s$ and $E_s = \omega_s|\tilde{A}_s|^2/2\pi c^2$, so that once again we have the standard relations: $J_s = E_s/\omega_s$ and $\Gamma_s = v_{gs}J_s$.

In conclusion, we point out that the success of our approach for deriving local Manley-Rowe relations, valid for noneikonal wave fields, requires that a Lagrangian formulation for the problem under consideration be available. This then implies that our nonlinear evolution equations must not contain any dissipation. Using the Whitham-average and Noether methods, the Manley-Rowe relations, with wave-action density and wave-action flux defined in (12), are derived in an efficient and straightforward fashion once the Noether symmetries are identified. We noted above that a complex representation for the wave fields was essential for obtaining nonvanishing expressions for the wave-action density and wave-action flux. Moreover, these expressions were shown to have the correct eikonal limit.

As an explicit example, we have considered a one-dimensional model for stimulated Raman scattering in an unmagnetized electron plasma. Expressions (16)–(19) were obtained for the Langmuir-wave and transverse-wave action densities and action fluxes without the need for the eikonal representation for the wave fields. These results can be generalized, e.g., by introducing the specific entropy s as a dynamical variable—the specific energy $\varepsilon(n,s)$ is then a function of n and s—or by considering a three-dimensional SRS model. Lagrangian formulations are available for both cases [8], and new local Manley-Rowe relations can be derived by the Noether method. A Lagrangian formulation for the stimulated Brillouin scattering process, involving the interaction of two transverse electromagnetic waves and an ion-acoustic wave, is also available, and local Manley-Rowe relations can again be derived by the Noether method.

This work was supported by the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

[1] V. N. Oraevsky, in *Handbook of Plasma Physics, Vol. 2: Basic Plasma Physics II,* edited by A. A. Galeev and R. N. Sudan (North-Holland, Amsterdam, 1984), Chap. 5.2; K. Mima and K. Nishikawa, *ibid.,* Chap. 6.5.
[2] C. N. Lashmore-Davies, Plasma Phys. **17**, 281 (1975); L. Stenflo, Phys. Scr. **T50**, 15 (1994).
[3] G. B. Whitham, *Linear and Nonlinear Waves* (John Wiley, New York, 1974), Sec. 11.7.
[4] D. Gabor, J. IEE (London) **93**, 429 (1946).
[5] G. B. Whitham, J. Fluid Mech. **44**, 373 (1970).
[6] Ref. [3], Sec. 7.7.
[7] W. L. Kruer, *The Physics of Laser Plasma Interactions* (Addison-Wesley, Redwood City, 1988), Chap. 7.
[8] R. L. Seliger and G. B. Whitham, Proc. R. Soc. London A **305**, 1 (1968).