Fermi- and Non-Fermi-Liquid Behavior in the Anisotropic Multichannel Kondo Model: Bethe Ansatz Solution

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We solve the multichannel Kondo model with channel anisotropy using the Bethe ansatz method. The model generates energy scales, characterizing the neighborhoods of the various infrared fixed points, reflecting the structure of the symmetry breaking in the channel sector. The nature of these fixed points also depends on the magnitude of the impurity spin S. We present a detailed discussion for the two channel case and point out some possible non-Fermi-liquid behavior.

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Non-Fermi-liquid (NFL) behavior observed in some Ce and U alloys [1] has stimulated an intense study of many impurity models, in particular, of the multichannel Kondo model [2]:

$$H = H_0 + 2\sum_{a,b} \sum_{m=1}^{f} J_m \psi^\dagger_{a,m}(0)\, \boldsymbol{\sigma}_{ab} \psi_{b,m}(0) \cdot \mathbf{S}\,, \quad (1)$$

where $H_0 = -i\sum_{a;m}\int dx\, \psi^\dagger_{a,m}(x)\,\partial_x \psi_{a,m}(x)$. Here the field $\psi_{a,m}$ describes electrons with spin index $a = \pm 1/2$ and orbital channel *(flavor)* index m, $m = 1,\ldots,f$, and we chose to set $v_F = 1$. The operator $\mathbf{S}$ represents the spin-S impurity localized at $x = 0$.

The infrared behavior of the model depends on the structure of the flavor sector [2]. In particular, a new behavior appears when $f > 2S$, with the overscreened system exhibiting NFL physics.

An exact solution was found for the isotropic case, $J_m = J$ [3,4]. The solution yields the spectrum and the thermodynamics for any temperature and magnetic field. In particular, the residual entropy and the critical exponents governing the low-energy physics were calculated. The thermodynamics was studied numerically in [5].

The neighborhood of the isotropic fixed point was further studied by means of conformal field theory [6] and bosonization methods [7], yielding the long distance asymptotics of the correlation functions with critical exponents that are the same as those characterizing the thermodynamic functions.

In this paper we present the solution of the channel-anisotropic model. We shall discuss in detail the case of two channels and arbitrary spin S, and briefly outline the generalization for more flavors.

We shall find that in the two channel case the model generates two scales, T_i and T_a, which we shall later interpret as associated with two fixed points, the isotropic and anisotropic, respectively. The scales are explicitly given (in our cutoff scheme) by $T_i \equiv De^{-\pi/J_1}$, $T_a \equiv D\cos[(J_1/J_2)\,\pi/2]\,e^{-\pi/J_2}$, where $D = N/L$. Here N is the number of electrons in each channel, L is the length of the system, and $J_1 \leq J_2$. In the scaling limit D is taken to infinity with the (bare) couplings' dependence on D chosen so as to keep the scales finite. The functional dependence of the scales on the coupling constants J_1 and J_2 is not universal and may change with the cutoff procedure. However, the dependence of physical quantities on the scales is universal. The ratio $\Delta \equiv T_a/T_i$ is the physical measure of the anisotropy.

The presence of flavor allows up to f electrons to interact simultaneously with the impurity. Therefore, spin composites of electrons form irrespectively of the degree of anisotropy, and their binding energy is set by the smallest of the couplings. The Hamiltonian must be regularized with care to allow the formation of the composites while maintaining integrability. We choose the regularization scheme used in [3], leading to the (regularized) first quantized form of the Hamiltonian (1),

$$h = \sum_{j=1}^{2N}[-i\partial_j - \Lambda^{-1}(\partial_j)^2 + 2J_j\delta(x_j)\,\boldsymbol{\sigma}_j \cdot \mathbf{S}]\,. \quad (2)$$

Here J_j is either J_1 or J_2. The limit $\Lambda \to \infty$ will be taken only after determining the eigenvalues. This limiting procedure, the *fusion* [3], leads to the formation of the spin composites. We shall find that the binding energy will be well above the spin energy scales, and the flavor excitations will disappear from the low-energy spectrum.

The eigenfunctions of (2) are combinations of plane waves with pseudomomenta $\{k_j, j = 1,\ldots,2N\}$ and amplitudes $A^{m_1\ldots m_{2N}}_{a_1\ldots a_{2N};\alpha}$ depending on the electron spin and flavor indices a_j, m_j, and the impurity spin index α. The energy eigenvalues in terms of the pseudomomenta are $E = \sum_{j=1}^{2N} k_j(1 + k_j/\Lambda)$, while the amplitudes are determined from the two body S matrices, to which we now turn.

The impurity-electron S matrix is derived from (2),

$$S_{j0} = \frac{\tilde{\lambda}_j + 1 - iJ_j(\boldsymbol{\sigma}\cdot\mathbf{S} + \frac{1}{2})}{\tilde{\lambda}_j + 1 - iJ_j}\,, \quad (3)$$

where $\tilde{\lambda}_j = k_j/\Lambda$. Since the interactions are flavor preserving, the S matrix (3) has only a nontrivial term in spin space.

All electrons are rightmovers, and as there is no direct interaction between them the electronic states are infinitely degenerate away from the impurity. This permits the introduction of an arbitrary electron-electron S matrix S_{jl} into the definition of the wave functions. To construct a basis that will manifest the integrability of the model the matrices S_{jl} must be chosen so as to satisfy the Yang-Baxter factorization equations,

$$S_{j0}S_{l0}S_{jl} = S_{jl}S_{l0}S_{j0}, \quad S_{jk}S_{lk}S_{jl} = S_{jl}S_{lk}S_{jk}\,. \qquad (4)$$

The solution for S_{jl} can be given as a direct product of spin and flavor terms, $S_{jl} = S_{jl}^{\text{spin}} \otimes S_{jl}^{\text{flavor}}$, with each term satisfying (4) separately. The spin component of S_{jl} is given by

$$S_{jl}^{\text{spin}} = \frac{(\lambda_j + 1/J_j) - (\lambda_l + 1/J_l) - iP_{jl}}{(\lambda_j + 1/J_j) - (\lambda_l + 1/J_l) - i}\,, \qquad (5)$$

where $\lambda_j = \tilde{\lambda}_j/J_j$ and $P_{jl} = \frac{1}{2}(\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_l + 1)$. The form of (5) reflects the $SU(2)_{\text{spin}}$ invariance of the model. The flavor component of S_{jl} reflects the breaking of the $SU(2)_{\text{flavor}}$ symmetry to a residual U(1) when the anisotropy is present, and is given by [8]

$$S_{jl}^{\text{flavor}} = \frac{i}{2}\frac{\sin\nu}{\sinh[\kappa(\tilde{\lambda}_j - \tilde{\lambda}_l) + i\nu]}\{\tau_x \otimes \tau_x + \tau_y \otimes \tau_y\} + \frac{1}{2}\{1 + \tau_z \otimes \tau_z\} + \frac{1}{2}\frac{\sinh[\kappa(\tilde{\lambda}_j - \tilde{\lambda}_l)]}{\sinh[\kappa(\tilde{\lambda}_j - \tilde{\lambda}_l) + i\nu]}\{1 - \tau_z \otimes \tau_z\}.$$

Here, $\{\tau\}$ are the Pauli matrices, and κ and ν are functions of the couplings. Denoting $\mu \equiv \nu/\kappa$, we shall see that μ is related to the binding energy of the composites.

Given the set of consistent S matrices we may derive the Bethe ansatz equations determining the allowed pseudomomenta k_j and hence the spectrum. Introducing the auxiliary variables $\{\omega_\gamma\}$ for the flavor sector, and $\{\chi_\gamma\}$ for the spin sector, we find

$$e^{ik_jL} = \prod_{\delta=1}^{M}\frac{\chi_\delta - \lambda_j - 1/J_j + i/2}{\chi_\delta - \lambda_j - 1/J_j - i/2}\prod_{\delta=1}^{M_1}\frac{\sinh[\nu(\omega_\delta - \tilde{\lambda}_j/\mu + i/2)]}{\sinh[\nu(\omega_\delta - \tilde{\lambda}_j/\mu - i/2)]}\,,$$

$$\prod_{j=1}^{2N}\frac{\sinh[\nu(\omega_\gamma - \tilde{\lambda}_j/\mu + i/2)]}{\sinh[\nu(\omega_\gamma - \tilde{\lambda}_j/\mu - i/2)]} = -\prod_{\delta=1}^{M_1}\frac{\sinh[\nu(\omega_\gamma - \omega_\delta + i)]}{\sinh[\nu(\omega_\gamma - \omega_\delta - i)]}\,,$$

$$-\prod_{\delta=1}^{M}\frac{\chi_\gamma - \chi_\delta + i}{\chi_\gamma - \chi_\delta - i} = \frac{\chi_\gamma + iS}{\chi_\gamma - iS}\prod_{j=1}^{2N}\frac{\chi_\gamma - \lambda_j - 1/J_j + i/2}{\chi_\gamma - \lambda_j - 1/J_j - i/2}\,,$$

describing the full content of the model.

The ground state and low lying energy excitations consist of solutions in the form of *double 2-strings,*

$$\lambda_\delta^{+\pm} = \frac{\mu}{J_2}\left(\omega_\delta \pm \frac{i}{2}\right), \quad \lambda_\gamma^{-\pm} = \frac{\mu}{J_1}\left(\omega_\gamma \pm \frac{i}{2}\right). \qquad (6)$$

The structure of the double string solution reflects the flavor symmetry breaking. In the isotropic limit the two strings coalesce leading to the SU(2) flavor degeneracy and the NFL behavior.

The energy associated with the double 2-string is $\varepsilon_\delta = 2\mu\omega_\delta(1 + \mu\omega_\delta/\Lambda) - \mu^2\Lambda/2$, leading to the *1-string* hypothesis for the $\{\omega_\gamma\}$, namely, $\omega_\delta = p_\delta/\mu\Lambda$ [3]. We substitute the *double* 2-string solution (6) in the eigenvalue equations and take the $\Lambda \to \infty$ limit. This is the *fusion* process. We find that the spin contribution to the energy is $E = \sum_{\delta=1}^{N} 2p_\delta$, and the spin degrees of freedom are described by the following system of fused Bethe ansatz equations:

$$-\prod_{\delta=1}^{M}\frac{\chi_\gamma - \chi_\delta + i}{\chi_\gamma - \chi_\delta - i} = \frac{\chi_\gamma + iS}{\chi_\gamma - iS}\left[\left(\frac{\chi_\gamma - 1/J_2 + \frac{i}{2}(1+\phi)}{\chi_\gamma - 1/J_2 - \frac{i}{2}(1+\phi)}\right)\left(\frac{\chi_\gamma - 1/J_2 + \frac{i}{2}(1-\phi)}{\chi_\gamma - 1/J_2 - \frac{i}{2}(1-\phi)}\right) \times \left(\frac{\chi_\gamma - 1/J_2 + \frac{i}{2}(1+\varphi)}{\chi_\gamma - 1/J_2 - \frac{i}{2}(1+\varphi)}\right)\left(\frac{\chi_\gamma - 1/J_2 + \frac{i}{2}(1-\varphi)}{\chi_\gamma - 1/J_2 - \frac{i}{2}(1-\varphi)}\right)\right]^{N/2},$$

with $\phi \equiv \mu/J_2$ and $\varphi \equiv \mu/J_1$. As the binding energy is set by the weakest coupling, μ is related to J_1, and in the scaling limit we find $\mu = J_1$, $\phi = J_1/J_2$, and $\varphi = 1$. Note that the equations reduce to the isotropic equations for $\phi = J_1/J_2 = 1$ and to the one-channel Kondo equations for $\phi = 0$, $J_1 = 0$.

We now solve the equations, identify the ground state and excitations, and, summing over the latter, derive the free energy. The main results are the following:

(i) The solutions of the equations are of the form given by the *string hypothesis* valid in the thermodynamic limit, $\chi_\delta^{n,k} = \chi_\delta^n + \frac{i}{2}(n + 1 - 2k)$, $k = 1, \ldots, n$, χ_δ^n real.

(ii) The ground state is composed of χ 1- and 2-strings, interpolating between the isotropic model (where $\phi = 1$ and the ground state built of 2-strings leading to NFL physics), and the single channel model [where $\phi = 0$ and the ground state is built of 1-strings describing a Fermi liquid (FL)].

(iii) The impurity free energy for impurity spin S, anisotropy Δ, temperature T, and magnetic field h is

$$F^i(\Delta, S; T, h) = -\frac{T}{2\pi}\int_{-\infty}^{\infty} d\xi\,\frac{\ln[1 + \eta_{2S}(\xi, h/T)]}{\cosh[\xi + \ln(T/T_i)]}\,.$$

The function $\eta_{2S}(\xi, h/T)$ belongs to the set of functions $\{\eta_n\}$ satisfying the following set of coupled integral equations, here written in the scaling limit ($D \to \infty$, keeping T_i, T_a fixed):

$$\ln\eta_1 = -2\Delta e^{\xi} + G\ln(1+\eta_2),$$
$$\ln\eta_2 = -e^{\xi} + G\ln(1+\eta_1) + G\ln(1+\eta_3),$$
$$\ln\eta_n = G\ln(1+\eta_{n-1}) + G\ln(1+\eta_{n+1}), \quad n > 2,$$

with boundary conditions $\lim_{n\to\infty}([n+1]\ln(1+\eta_n) - [n]\ln(1+\eta_{n+1})) = -2\mu h/T$. The integral operators $[\alpha]$ and G are defined by the kernels $\alpha/2\pi[(\xi'-\xi)^2 + (\alpha/2)^2]$ and $1/2\cosh[\pi(\xi-\xi')]$, respectively.

We now elaborate on (iii). We studied analytically the equations in the limits of small and large anisotropy, both for $S = 1/2$ and for $S = 1$, and deduced the infrared behavior of the thermodynamic functions and its dependence on the anisotropy Δ.

The $S = 1/2$ impurity contribution to the free energy at low temperatures is given by

$$F^i_{S=1/2} \sim \begin{cases} -\frac{1}{2}[(\frac{1}{2}\gamma_{a,1/2}\frac{1}{\Delta} - \gamma_{i,1/2}\ln\Delta) - \omega_{i,1/2}(\frac{h}{T})^2\ln\Delta]\frac{T^2}{T_i}, & T \ll T_a = T_i \quad \Delta \ll 1 \\ -\frac{T}{2}\ln 2 + \frac{1}{2}[\gamma_{i,1/2} + \omega_{i,1/2}(\frac{h}{T})^2]\frac{T^2}{T_i}\ln\frac{T}{T_i} + \mathcal{O}(\frac{T^2}{T_i}) + \mathcal{O}(\Delta\frac{T^2}{T_i}), & T_a \ll T \ll T_i \quad \Delta \ll 1 \\ -\frac{1}{2}(1-\frac{\delta}{\Delta})[\gamma_{a,1/2} + \omega_{a,1/2}(\frac{h}{T})^2 + \mathcal{O}(\Delta e^{-\Delta})]\frac{T^2}{T_a}, & T \ll \frac{T_a}{\Delta}, \quad \Delta \gg 1 \end{cases}$$

where $\gamma_{a,1/2} = \pi/6$, $\omega_{a,1/2} = 1/\pi$ are the coefficients for the specific heat and susceptibility of the $S = 1/2$ single channel Kondo model, $\gamma_{i,1/2} = \pi/4$, $\omega_{i,1/2} = 2/\pi$ are those of the $S = 1/2$ two-channel isotropic model, and δ is a constant close to 1.

The free energy yields the susceptibility and the specific heat. We find, for any $\Delta > 0$, a linear specific heat and a temperature independent susceptibility. Therefore, any small amount of anisotropy moves the system away from the isotropic fixed point. In particular, the $\frac{1}{2}\ln 2$ contribution to the $T = 0$ entropy that appears in the isotropic case [3] is no longer present. However, for small anisotropy the behavior of the system in the temperature range $T_a = \Delta T_i < T < T_i$ is identical to that of the isotropic case, and the isotropic NFL behavior reemerges upon setting $\Delta = 0$. For large anisotropy, on the other hand, the leading terms of the thermodynamic functions are those of the $S = 1/2$ single channel model with T_a playing the role of the Kondo temperature.

Computing the ratio R,

$$R_{1/2} = \frac{\chi^i/\chi^e}{C^i_v/C^e_v} \sim \begin{cases} \frac{8}{3}, & \Delta = 0, \\ -8\Delta\ln\Delta, & 0 < \Delta \ll 1, \\ 2 + \mathcal{O}(\Delta e^{-\Delta}), & \Delta \gg 1, \end{cases}$$

we again conclude that turning on the anisotropy destroys the NFL behavior of the isotropic model, while the $\Delta = \infty$ single channel FL is only weakly modified upon reducing the anisotropy.

We now turn to $S = 1$. The impurity free energy and R have the following low-T expressions:

$$F^i_{S=1} \sim \begin{cases} -\frac{1}{2}\{[\gamma_{i,1} + \omega_{i,1}(\frac{h}{T})^2] - [\alpha + \beta(\frac{h}{T})^2\Delta\ln\Delta]e^{-1/\Delta}\}\frac{T^2}{T_i}, & T \ll \Delta T_i, \quad \Delta \ll 1 \\ -\{(1 + \frac{\ln\Delta}{\Delta})[\gamma_{a,1/2} + \omega_{a,1/2}(\frac{h}{T})^2] + \mathcal{O}(e^{-\Delta})\}\frac{T^2}{T_i}, & T \ll T_i, \quad \Delta \gg 1 \\ -[\alpha' + \beta'(\frac{h}{T})^2\frac{1}{\Delta}]\frac{T^2}{T_i} - (\frac{h}{T})^2\frac{cT}{\ln T/T_i}, & T_i \ll T \ll \Delta T_i \quad \Delta \gg 1 \end{cases}$$

$$R_1 \sim \begin{cases} \frac{8}{3} + \mathcal{O}(e^{-1/\Delta}), & \Delta \ll 1, \\ 2 + \mathcal{O}(e^{-\Delta}), & \Delta \gg 1, \end{cases}$$

where now α, α', β, β', and c are constants of order 1, and $\gamma_{i,1} = \pi/2$ and $\omega_{i,1} = 4/\pi$ are the coefficients for the specific heat and the susceptibility of the $S = 1$ isotropic model.

When $T_i = 0$ and T_a is finite, the system behaves as a single channel Kondo model with $S = 1$, describing at low temperatures a partially screened spin. As J_1 is turned on ($\Delta \gg 1$), the system undergoes another Kondo screening, as can be seen in the temperature range $T_i \ll T \ll \Delta T_i = T_a$, and ends up in the infrared as a combination of two screened $S = 1/2$ single channel models.

We also carried out a numerical solution over the whole range of anisotropy and temperature (Fig. 1). Note (e.g., in the spin-1/2 case) the two stage quenching of the ln2 high-temperature entropy when $T_a \leq T_i$. For $T_a > T_i$ the quenching occurs in one step at T_a. Similarly, the two peaks present in the specific heat for $\Delta \leq 1$ merge and move with T_a as Δ is increased past 1. Related observations apply also in the spin-1 case as Δ is reduced past 1.

We turn now to interpret our results in terms of fixed-point Hamiltonians. Clearly, we reach two different fixed points, the isotropic i and the anisotropic a when we set $\Delta = 0$ and $\Delta = \infty$, respectively. Both fixed points have directions of instability: in the screened case a is unstable to turning on J_1 ($T_i > 0$), while in the overscreened case i is unstable to turning on the anisotropy ($T_a > 0$). Where do the trajectories flow to? The evidence thus far, while consistent with i being the end point in the former case and a in the latter, also suggests the exciting possibility (with several phenomenological implications) of the trajectories ending on lines of fixed points labeled by Δ, with FL behavior in the screened and NFL behavior in the overscreened case [9]. The anisotropy dependence of the

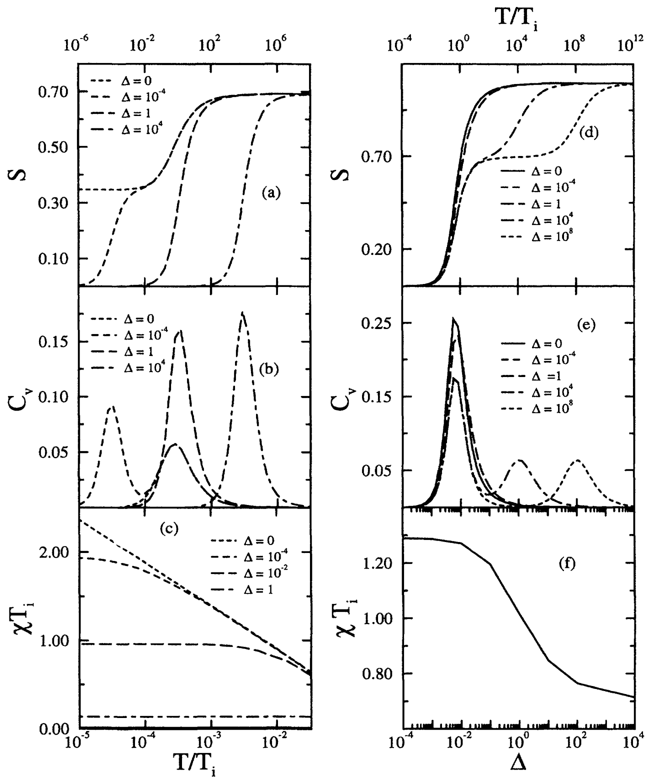

FIG. 1. (a) Entropy, (b) specific heat, and (c) zero field susceptibility for $S = 1/2$ as a function of T/T_i, for different values of the anisotropy parameter $\Delta = T_a/T_i$. (d) and (e) Same as (a) and (b), now for $S = 1$. (f) Zero field susceptibility at $T = 0$ as a function of the anisotropy parameter Δ.

thermodynamic coefficients is then due to marginal rather than irrelevant operators. To completely resolve the issue it is necessary to calculate the asymptotics of the correlation functions. This work is in progress.

The structure of the thermodynamic equations reflects the flavor symmetry breaking and generalizes to any number of flavors. The various patterns of the SU(f)-symmetry breaking and their relative strengths will be parametrized by energy scales $T_n = Dg_n(J_1, \ldots, J_f)$, $\alpha = 1, \ldots, f$. The scales set the excitation energies and momenta and appear in the thermodynamic equations,

$$\ln\eta_n = \begin{cases} -\frac{T_n}{T} e^{\xi} + G\ln(1 + \eta_{n-1}) \\ \quad +G\ln(1 + \eta_{n+1}), & n \leq f, \\ G\ln(1 + \eta_{n-1}) + G\ln(1 + \eta_{n+1}), & n > f, \end{cases}$$

with $\eta_0 \equiv 0$. As an illustration we discuss the three-channel problem, with scales T_1, T_2, and T_3. The isotropic case is characterized by $T_3 > 0$, $T_1 = T_2 = 0$. The low-energy physics is NFL for $S < 3/2$ with $C_v \sim T^{4/5}$ and zero-temperature entropy $S = \ln[\sin(2S + 1)\pi/5]/(\sin\pi/5)$. When the symmetry is broken to SU(2) × U(1), a new scale appears. When $T_1 > 0$ and $T_2 = 0$, the models with $S = 1/2$ and $S = 3/2$ show linear T dependence in C_v, and constant χ, with different coefficients for the $T_1 > T_3$ and $T_1 < T_3$ regions. The $S = 1$ case is more interesting, since there is a change from linear T dependence to NFL behavior, and a residual $T = 0$ entropy emerges as the parameters of the system are changed from $T_1 < T_3$ to $T_3 < T_1$. When $T_1 = 0$, $T_2 > 2$, and $S = 1/2$, we have two different NFL fixed points for $T_2 < T_3$ and $T_3 < T_2$ with different values of the residual entropy; the $S = 1, 3/2$ cases have constant C_v/T and χ. Finally, when all the scales are finite, the system has constant C_v/T and χ for $S = 1/2, 1, 3/2$, with the coefficients depending on the relation between the T_i and with intermediate temperature regions where the properties of the thermodynamic quantities correspond to those described in the previous cases. These considerations generalize for any number of flavors f.

In forthcoming work we study the competition between strength of the couplings and the size of symmetry breaking. We also consider the role of the generalized fusion mechanism in the two-impurity Kondo model, and in models [e.g., with O(N) symmetry] giving rise to new types of fixed points. Finally, we shall discuss some phenomenological implications.

We received two preprints [10] addressing the model by means of bosonization and the Anderson-Yuval approach. While there is a considerable overlap with our work, there are also interesting differences. For example, the crossover from a two-scale to a one-scale dynamics at $\Delta = 1$ is not seen.

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