New Cross Term in the Two-Particle Hanbury-Brown–Twiss Correlation Function in Ultrarelativistic Heavy-Ion Collisions

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Using two specific models and a model-independent formalism, we show that in addition to the usual quadratic "side," "out," and "longitudinal" terms, a previously neglected "out-longitudinal" cross term arises naturally in the exponent of the two-particle correlator. Since its effects can be easily observed, such a term should be included in any experimental fits to correlation data. We also suggest a method of organizing correlation data using rapidity rather than longitudinal momentum differences, since in the former every relevant quantity is longitudinally boost invariant.

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The experimentally measured Hanbury-Brown–Twiss (HBT) correlation between two identical particles emitted in a high energy collision defines a six-dimensional function of the momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ [1]. A popular way of presenting these is in terms "size parameters" derived from a Gaussian fit to the data of the form [2–5]

$$C(\mathbf{q},\mathbf{K}) = 1 \pm \lambda \exp[-q_s^2 R_s^2(\mathbf{K}) - q_o^2 R_o^2(\mathbf{K}) - q_l^2 R_l^2(\mathbf{K})], \tag{1}$$

where $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2$, $\mathbf{K} = \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)$, the + (−) sign is for bosons (fermions), and the HBT Cartesian coordinate system is defined as follows [6]: The "longitudinal" or $\hat{z}$ (subscript l) direction is parallel to the beam, the "out" or $\hat{x}$ (subscript o) direction is parallel to the component of $\mathbf{K}$ which is perpendicular to the beam, and the "side" or $\hat{y}$ (subscript s) direction is the remaining direction.

In this Letter we assert that significantly more can be learned and better fits achieved if an "out-longitudinal" cross term is included in any Gaussian fits to the data. In other words, we suggest that the data should be fitted by a function with the following form:

$$C(\mathbf{q},\mathbf{K}) = 1 \pm \lambda \exp[-q_s^2 R_s^2(\mathbf{K}) - q_o^2 R_o^2(\mathbf{K}) - q_l^2 R_l^2(\mathbf{K}) - 2q_o q_l R_{ol}^2(\mathbf{K}), \tag{2}$$

where R_{ol}^2 is a parameter which can be either positive or negative; we simply use the R^2 notation to denote the fact that it is has the dimension of an area.

Since particles 1 and 2 are indistinguishable, the overall sign of $\mathbf{q}$ is irrelevant. The relative signs of the various components of $\mathbf{q}$, however, are well-defined physical quantities for any given pair. Our sign convention for $\mathbf{q}$ will be such that q_s is always positive. We can thus unambiguously discuss correlations for negative as well as positive values of both q_o and q_l.

To see how an out-longitudinal cross term arises in two-particle correlations, we use the following well-established theoretical approximation [7,8]:

$$C(\mathbf{q},\mathbf{K}) \simeq 1 \pm \frac{|\int d^4x\, S(x,K) e^{iq\cdot x}|^2}{|\int d^4x\, S(x,K)|^2}, \tag{3}$$

where $q_0 = E_1 - E_2$, $K_0 = E_K = \sqrt{m^2 + |\mathbf{K}|^2}$. Here $S(x,K)$ is a function which describes the phase space density of the emitting source. For pairs with $|\mathbf{q}| \ll E_K$,

$$q\cdot x \simeq (\beta_o q_o + \beta_l q_l)t - q_o x - q_s y - q_l z, \tag{4}$$

where $\beta_i = K_i/E_K$.

As a simple example, we consider the following cylindrically symmetric Gaussian emission function:

$$S(x,K) = f(K) \exp\left[-\frac{x^2+y^2}{2R^2} - \frac{z^2}{2L^2} - \frac{(t-t_0)^2}{2(\delta t)^2}\right]. \tag{5}$$

Using (3) and (4), it is easy to see that the corresponding correlation function takes the form

$$C(\mathbf{q},\mathbf{K}) = 1 \pm \exp\{-q_s^2 R^2 - q_o^2[R^2 + \beta_o^2(\delta t)^2] - q_l^2[L^2 + \beta_l^2(\delta t)^2] - 2q_o q_l \beta_o \beta_l (\delta t)^2\}. \tag{6}$$

For this model, the $q_o q_l$ cross term thus provides a measurement of the duration of particle emission (δt).

One might think that this cross term is just a trivial kinematic effect which would not arise if the correlation were calculated in some more carefully chosen coordinate system or reference frame. For example, for spherically symmetric systems ($L = R$), it has been shown that the cross term vanishes if axes are chosen parallel and perpendicular to $\mathbf{K}$ (rather than parallel and perpendicular to the beam) [9]. The reader can verify, however, that for systems with $L \neq R$ the cross term does not vanish in these rotated coordinates, but rather measures the $L^2 - R^2$ asymmetry of the source.

Another system that is often proposed is the longitudinally comoving system (LCMS), which is defined as the frame in which $\beta_l = 0$ [4,5,10]. Glancing at (6), it naively appears that the cross term will vanish in this frame. Being more careful, however, one can see that after transforming to the LCMS frame (primed variables)

$$t' = \gamma_l(t - \beta_l z), \qquad z' = \gamma_l(z - \beta_l t), \tag{7}$$

where $\gamma_l = 1/\sqrt{1-\beta_l^2}$, $t'z'$ terms arise in the transformed emission function S'. These in turn lead to a nonvanishing cross term of the form

$$R_{ol}'^2 = \beta_o \beta_l \gamma_l^2 [(\delta t)^2 + L^2], \tag{8}$$

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where β_l and γ_l are evaluated in the center of mass frame. We can see that the cross term cannot, in general, be removed simply by transforming to another coordinate system or reference frame, and therefore it is certainly not just a trivial kinematic effect. In fact, R_{ol}^2 contains physical information about the emitting source which is just as important as that found from evaluating the difference $R_o^2 - R_s^2$.

In order to get a broader feeling for what the cross term measures, we now introduce a general formalism valid for any cylindrically symmetric emission function which can be expressed in a roughly Gaussian form. Using (4), we expand $\exp(iq\cdot x)$ in (3) for $q\cdot x \ll 1$ to find (see [11])

$$C(\mathbf{q},\mathbf{K}) = 1 \pm \{1 - q_s^2\langle y^2\rangle - \langle[q_o(\beta_o t - x) + q_l(\beta_l t - z)]^2\rangle + \langle q_o(\beta_o t - x) + q_l(\beta_l t - z)\rangle^2 + \mathcal{O}[(q\cdot x)^4]\}, \quad (9)$$

where the $q_s q_o$ and $q_s q_l$ terms vanish due to cylindrical symmetry [8] and we have introduced the notation

$$\langle\xi\rangle \equiv \langle\xi\rangle(\mathbf{K}) = \frac{\int d^4x\, \xi\, S(x,K)}{\int d^4x\, S(x,K)}. \quad (10)$$

Exponentiating (9), we see that for any cylindrically symmetric system the correlation function for small momentum differences ($q_i R_i \ll 1$) can be expressed in the form of Eq. (2), where $\lambda = 1$ and the functions $R_i^2(\mathbf{K})$ can simply be read off as the coefficients of the corresponding $q_i q_j$ terms in Eq. (9):

$$\begin{aligned} R_s^2 &= \langle y^2\rangle - \langle y\rangle^2 = \langle y^2\rangle, \\ R_o^2 &= \langle(x - \beta_o t)^2\rangle - \langle x - \beta_o t\rangle^2, \\ R_l^2 &= \langle(z - \beta_l t)^2\rangle - \langle z - \beta_l t\rangle^2, \\ R_{ol}^2 &= \langle(x - \beta_o t)(z - \beta_l t)\rangle - \langle x - \beta_o t\rangle\langle z - \beta_l t\rangle. \quad (11) \end{aligned}$$

These expressions have the typical form of variances and show that the HBT "size parameters" are really lengths of homogeneity associated with the source function $S(x,K)$ [12]. For a static source, these homogeneity lengths are equal to its geometric size in the various directions which can then be directly extracted from the HBT correlator. For expanding sources the interpretation of the HBT size parameters is more involved [7–10,12,13], and the HBT parameters are usually smaller than the geometric extensions of the source. The cross term is seen to measure the temporal extent of the source as well as the xz, xt, and zt correlations of the emission function. Note that the LCMS radii can be found from (11) by setting $\beta_l = 0$ and using $S(x',K')$ [see Eq. (7)]. The cross term vanishes in this frame if and only if the source $S(x',K')$ is reflection symmetric under $z' \longrightarrow -z'$ [which is not the case for our source (5)].

One might argue that the model independent expressions of Eq. (11) should not be compared to experimental correlation radii, since the former measure second derivatives of the correlation function around $\mathbf{q} = 0$ (because we used $q\cdot x \ll 1$ to derive them), while the latter are parameters of a Gaussian fit by the whole correlation function [2–5]. On the other hand, for any source which has a roughly Gaussian profile in some complete set of spatial coordinates, the two different methods of measuring radii will give roughly the same results. For these types of models, the simple expressions generated by Eq. (11) provide valuable insights as to how various parameters of the source distribution qualitatively affect measurable features of the correlation function.

We already discussed one Gaussian model in Eq. (5), but here we would like to discuss another, possibly more realistic, model which is similar to the ones presented in [13]. In the center of mass frame of an expanding fireball, we define the following emission function:

$$S(x,K) = \frac{\tau_0\, m_t \cosh(\eta - Y)}{(2\pi)^3 \tau\sqrt{2\pi(\delta\tau)^2}} \exp\left[-\frac{K\cdot u(x)}{T}\right] \times \exp\left[-\frac{\rho^2}{2R_G^2} - \frac{\eta^2}{2(\delta\eta)^2} - \frac{(\tau - \tau_0)^2}{2(\delta\tau)^2}\right], \quad (12)$$

where T is a constant freeze-out temperature, $\rho = \sqrt{x^2 + y^2}$, $\tau = \sqrt{t^2 - z^2}$, $\eta = \frac{1}{2}\ln[(t+z)/(t-z)]$, $m_t = \sqrt{m^2 + K_\perp^2}$, and Y is the rapidity of a particle with momentum $\mathbf{K}$. Note that in the limit $\delta\tau \longrightarrow 0$, (12) becomes the Boltzmann approximation for a hydrodynamic system with local flow velocity $u(x)$ which freezes out on a three-dimensional hypersurface of constant longitudinal proper time τ_0 and temperature T [8,14]. We will consider a flow which is nonrelativistic tranversally but which exhibits Bjorken expansion longitudinally

$$u(x) \simeq ([1 + \tfrac{1}{2}(v\rho/R_G)^2]\cosh\eta,\ vx/R_G,\ vy/R_G,\ [1 + \tfrac{1}{2}(v\rho/R_G)^2]\sinh\eta), \quad (13)$$

where $v \ll 1$ is the transverse flow velocity of the fluid at $\rho = R_G$. In [8] we show that when the emission function (12) is integrated over spacetime, it produces a very reasonable one-particle distribution.

To facilitate calculating the correlation function, we can make the physically reasonable assumption that $\delta\tau/\tau_0 \lesssim \frac{1}{2}$ so that we can be justified in replacing integrals over only positive values of τ with ones ranging from $-\infty$ to $+\infty$. We can then achieve analytic results by making a Taylor expansion [8] in the parameter

$$\frac{1}{(\delta\eta)_*^2} = \frac{1}{(\delta\eta)^2} + \frac{m_t}{T}. \quad (14)$$

Note that for pairs in which $m_t/T \gg 1/(\delta\eta)^2$ as were studied in [15], $(\delta\eta)_*^2$ becomes simply T/m_t.

Using the expressions (11) and keeping a few subleading corrections which are important when considering pions [8], we find

$$R_s^2 = R_*^2, \qquad R_o^2 = R_*^2 + \frac{K_\perp^2}{m_t^2}(\delta\tau)^2 + \frac{K_\perp^2}{m_t^2}\beta_l^2\tau_0^2(\delta\eta)_*^2 + \frac{K_\perp^2}{m_t^2}\left[1 + \beta_l^2 - 2\beta_l\frac{Y}{(\delta\eta)^2}\right](\delta\tau)^2(\delta\eta)_*^2$$
$$+ \frac{K_\perp^2}{m_t^2}\tau_0^2\left[\beta_l^2\nu - 2\beta_l\frac{Y}{(\delta\eta)^2} + \frac{1}{2}\right](\delta\eta)_*^4,$$
$$R_l^2 = \frac{m_t^2}{E_K^2}\tau_0^2(\delta\eta)_*^2 + \frac{m_t^2}{E_K^2}(\delta\tau)^2(\delta\eta)_*^2 + \frac{m_t^2}{E_K^2}\nu\tau_0^2(\delta\eta)_*^4,$$
$$R_{ol}^2 = -\beta_o\beta_l\tau_0^2(\delta\eta)_*^2 - \beta_o\left[\beta_l - \frac{Y}{(\delta\eta)^2}\right](\delta\tau)^2(\delta\eta)_*^2 - \beta_o\tau_0^2\left[\beta_l\nu - \frac{Y}{(\delta\eta)^2}\right](\delta\eta)_*^4, \qquad (15)$$

where

$$a\frac{1}{R_*^2} = \frac{1}{R_G^2}\left(1 + \frac{m_t}{T}v^2\right) \qquad (16)$$

and $\nu = 1 + (R_*/R_G)^2 - \frac{1}{2}(m_t/T)(\delta\eta)_*^2$. As pointed out in [13] and seen from Eq. (16), transverse flow causes the side radius to measure something smaller than the real geometrical radius R_G. What it does measure is the transverse region of homogeneity of the fluid as seen by particles with a given p_t [8,12]. It is also interesting that $R_o^2 - R_s^2$ depends on the average rapidity and is not quite directly proportional to the duration of particle emission $\delta\tau$ even for pairs with $\beta_l = 0$ [16]. In our opinion, however, the most interesting feature of this model is the cross term radius R_{ol}^2. Although just as in Eq. (6) the cross term vanishes for pairs with either $\beta_o = 0$ or $\beta_l = 0$, it will, in general, have an important effect on the correlation function, especially for pairs with large $|Y|$.

The effect becomes most easily apparent when we plug in some numbers and plot the correlation function. For simplicity, we consider a pion source with no transverse flow ($v = 0$) which freezes out instantaneously ($\delta\tau = 0$) with the following other source parameters: $R_G = 3$ fm, $\tau_0 = 4$ fm/c, $\delta\eta = 1.5$, and $T = 150$ MeV. Restricting ourselves to pairs with $Y = -2$, $K_\perp = 200$ MeV, and $q_s = 0$, we can now calculate the correlation function both by using the approximate analytic radii of (15) and by performing a numerical calculation directly from Eqs. (3) and (12). Comparing the results, we have found that the Gaussian approximation of (2) with the radii of (15) is able to describe the numerically calculated correlation function to within about 20% [8]. Figure 1 shows a plot of the latter as a function of q_o and q_l. The effect of the cross term can be seen in the form of an asymmetric ridge running from the peak at $q_o = q_l = 0$ down to the front left where $q_l > 0$ and $q_o < 0$. This kind of ridge is clearly identifiable experimentally and has, in fact, already been seen in preliminary NA35 correlation data [17].

Because of the boost invariance of the flow profile, the LCMS radii corresponding to the above model can be obtained simply by setting $\beta_l = 0$ and $E_K = m_t$ in (15). Note that the factor Y in the second line of R_{ol}^2 should not be set equal to zero, since it arises from the η distribution of the source (12) which obviously breaks the boost invariance of the emission function. It can be verified that if a source is *completely* boost invariant then the cross term will always vanish in the LCMS frame [15]. However, any source with a finite size ($\delta\eta < \infty$) cannot be completely boost invariant, so it will, in general, feature a nonvanishing cross term in the LCMS, since the LCMS does not coincide with the local rest frame [18].

We would now like to suggest a better way of organizing correlation data from sources undergoing boost-invariant longitudinal expansion [19]. Returning to (3), let us make an alternative on-shell definition of the four-vector K:

$$K = (m_t\cosh Y, \mathbf{K}_t, m_t\sinh Y), \qquad (17)$$

where $\mathbf{K}_t = \frac{1}{2}(\mathbf{p}_{1t} + \mathbf{p}_{2t})$, $m_t^2 = m^2 + |\mathbf{K}_t|^2$, $Y = \frac{1}{2}(\mathrm{y}_1 + \mathrm{y}_2)$, and y_i is the rapidity of the ith particle. The resulting approximation is at least as good as the approximation we have been using up to now [8]. This definition suggests that we express the correlation function in terms of q_s, q_o and the rapidity difference $\mathrm{y} = \mathrm{y}_1 - \mathrm{y}_2$:

$$C(\mathrm{y}, q_s, q_o Y, K_\perp) \simeq 1 \pm \lambda\exp[-q_s^2R_s^2 - q_o^2R_o^2 - \mathrm{y}^2\alpha^2 - 2q_o\mathrm{y}R_{o\mathrm{y}}]. \qquad (18)$$

The reader should take care not to confuse the rapidity difference y with the Cartesian coordinate y.

The model independent expressions corresponding to (11) are now given by

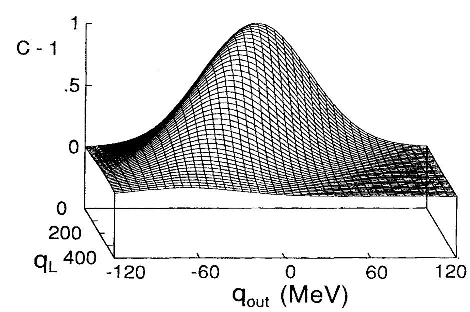

FIG. 1. The numerically calculated correlation function (3) generated by the pion source (12) with parameters $v = \delta\tau = 0$, $R_G = 3$ fm, $\tau_0 = 4$ fm/c, $\delta\eta = 1.5$, and $T = 150$ MeV is plotted as a function of q_l and q_o for $Y = -2$, $K_\perp = 200$ MeV, and $q_s = 0$.

$$R_s^2 = \langle y^2 \rangle, \qquad R_o^2 \langle [x - (K_\perp/m_t)\tau \cosh(\eta - Y)]^2 \rangle - \langle x - (K_\perp/m_t)\tau \cosh(\eta - Y) \rangle^2,$$
$$\alpha^2 = \langle [m_t \tau \sinh(\eta - Y)]^2 \rangle - \langle m_t \tau \sinh(\eta - Y) \rangle^2,$$
$$R_{oy} = \langle [m_t x - K_\perp \tau \cosh(\eta - Y)] \tau \sinh(\eta - Y) \rangle - \langle m_t x - K_\perp \tau \cosh(\eta - Y) \rangle \langle \tau \sinh(\eta - Y) \rangle. \qquad (19)$$

For the model (12), the radii take the much simpler form:

$$R_x^2 = R_*^2, \qquad R_o^2 = R_*^2 + \frac{K_\perp^2}{m_t^2}\{[1 + (\delta\eta)_*^2](\delta\tau)^2 + \tfrac{1}{2}(\delta\eta)_*^4 \tau_0^2\}, \qquad \alpha^2 = m_t^2 (\delta\eta)_*^2 \{\tau_0^2 [1 + \nu(\delta\eta)_*^2] + (\delta\tau)^2\},$$
$$R_{oy} = \frac{K_\perp Y}{(\delta\eta)^2}(\delta\eta)_*^2 [(\delta\eta)_*^2 \tau_0^2 + (\delta\tau)^2]. \qquad (20)$$

The astute reader will note that the above side and out radii are identical to the LCMS versions of (15), and that aside from a slight difference in the definition of Y, $R_l(\mathrm{LCMS}) = \alpha/m_t$ and $R_{ol}^2(\mathrm{LCMS}) = R_{oy}/m_t$. In fact, for systems undergoing Bjorken longitudinal expansion, LCMS correlation functions are nothing more than approximations to fixed frame correlation functions in rapidity coordinates [8].

Using the same source parameters as in Fig. 1, the effect of the cross term can be seen in Fig. 2 where we plot the correlator as a function of y for $q_o = 30$ MeV. The accuracy of the analytic approximation (dashed) is seen by comparing it with the exact numerical result (solid).

We have shown that an out-longitudinal (or "out-rapidity") cross term arises naturally both in a general Gaussian derivation of the correlation function and in two specific Gaussian models. Although transformed, in general, the cross term persists when one switches from calculating momentum differences in a fixed frame to calculating them in the LCMS. Consequently, there is no reason why such a term should be excluded *a priori* from Gaussian fits to experimental correlation data. Not only will the new parameter reveal more information about the source, its inclusion will undoubtably increase the accuracy of the other fitted radii.

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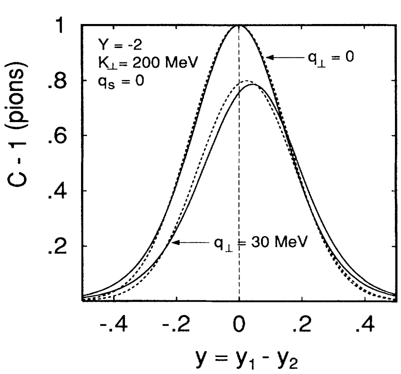

FIG. 2. The same parameters as in Fig. 1 are used to plot the correlation as a function of $y = y_1 - y_2$ for $q_o = 0$ (upper curves) and $q_o = 30$ MeV (lower curves) as calculated numerically (solid curves) and analytically via (15) (dashed curves).

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