Vectorial Interactions and Quantum Chaos

H. M. Fried
Physics Department, Brown University, Providence, Rhode Island 02912

Y. Gabellini
Institut Non-Linéaire de Nice, 1361 Route des Lucioles, 06560 Valbonne, France

B. H. J. McKellar
School of Physics, University of Melbourne, Parkville, Victoria, 3052 Austrailia
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Vectorial interactions (e.g., QED) introduce the possibility of chaos into new, nonperturbative Green's function representations of potential theory. The full, radiative corrections of quantum field theory, however, remove such chaos in an explicit example of "environment-induced decoherence."

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Causal, relativistic Green's functions $G_c(x,y\,|\,A)$ for the propagation of a particle of mass m in a specified field, or potential, $A(z)$, are used in many branches of physics and have two basic features: they may be written in terms of a proper-time integral $\int_0^\infty ds$, which acts, in part, to specify the causal nature of the function; and they contain integrals $\int d^4z$ over all space-time coordinates z_μ, which provide energy-momentum conservation at every potential interaction:

$$G_c(x,y\,|\,A) = i\int_0^\infty ds\, e^{-ism^2}\int d^4z\,\langle x|\mathcal{F}(s,z)|y\rangle . \quad (1)$$

Theories with scalar interactions require a Green's function which satisfies

$$[m^2 - \partial_x^2 + gA(x)]G_c(x,y\,|\,A) = \delta^4(x-y)$$

and have an exact Fradkin representation which may be conveniently written in momentum space in the form

$$\langle p|\mathcal{F}(s;z)|p'\rangle = e^{-iq\cdot z + isq^2/4}\prod_N{}' \frac{(-i)^2}{(2\pi)^4} \times \int\int d^4P_N\, d^4Q_N\, e^{i(P_N^2+Q_N^2)/2} \times e^{-isp^2}\exp\left\{-ig\int_0^s ds'\,A[\zeta(s')]\right\} \quad (2)$$

where $q = p - p'$, $\mathcal{P} = p + p'$, $\zeta(s') = z + s'\mathcal{P} - \sum_N' R_N(s')$, and

$$R_N(s') = \frac{2\sqrt{s}}{\pi N}\left[P_N\cos\left(\frac{N\pi s'}{s}\right) + Q_N\sin\left(\frac{N\pi s'}{s}\right)\right].$$

Here $\sum_N'$ and $\prod_N'$ correspond to the summation and product over terms specified by all positive, odd integers $N = 1,3,5,\ldots$. Equation (2) is an exact variant of Fradkin's functional representation [1] for $G_c[A]$, in which all quantum fluctuations [all terms of $\zeta(s') - z$ are $O(\hbar/mc)$] appear in the argument of a linear dependence on A.

Representation (2) is a fairly new result [2], which suggests a new, nonperturbative approach to such Green's functions by the simple expedient of limiting the quadratures associated with different N values to a finite few. (It should be noted that when any of the N-dependent fluctuations of R_N are neglected, the corresponding normalization integrals of Eq. (2) generate a factor of $+1$.) For such finite-N quadrature approximations, one has, in advance, a qualitative estimate of relative errors; and as shown in Ref. [2] by comparison with a soluble, nontrivial example, such estimates of relative error may be thought of as upper bounds. The existence of such a tractable, nonperturbative approach to $G_c[A]$ is the reason why Eq. (2) can be of considerable interest in a variety of fields.

When vectorial interactions are considered, for example, QED or QCD, a new feature appears [3] in these nonperturbative representations: the possibility of chaos. For simplicity, we here state the resulting Green's function for a spinless boson field interacting with a Maxwell field, according to the relation

$$\left\{m^2 - \sum_\mu[\partial_\mu^x - gA(x)]^2\right\}G_c(x,y\,|\,A) = \delta^4(x-y) .$$

Using techniques identical to those of Ref. [3], one constructs the representation [Eq. (1)] with the second line of Eq. (2) replaced by

$$\mathcal{N}'\int d[\phi]\int d[\Omega]\exp\left\{-i\int_0^s ds'[p_\mu - \Omega_\mu(s')]^2 + i\int_0^s ds'\,\phi_\mu(s')\left[\Omega_\mu(s') - gA_\mu\left(\zeta(s') - 2\int_0^{s'}\Omega\right)\right]\right\}, \quad (3)$$

where $\mathcal{N}'$ is a normalization factor which disappears upon performing the functional integration $\int d[\phi]$. The reason for this complication, relative to the scalar case, is that each component of A_μ must be allowed to transfer 4-momentum to the first quantized particle m in a well defined but nontrivial way. The result of the $\int d[\phi]$ is then to produce a delta functional $\delta[\Omega_\mu(s') - gA_\mu(\zeta(s') - 2\int_0^{s'}\Omega)]$ at each point $0 \leq s' \leq s$, which then allows the $\int d[\Omega]$ to be performed immediately, such that Eq. (3) is replaced by

$$\exp\left[-i\int_0^{s'} ds'[p_\mu - \Omega_\mu(s')]^2 - \mathrm{Tr}\ln[\delta f/\delta\Omega]\right],$$

where the function $\Omega_\mu(s')$ is to satisfy the "map"

$$\Omega_\mu(s') = gA\left(\zeta(s') - 2\int_0^{s'} ds''\,\Omega(s'')\right), \qquad (4)$$

and the Jacobian may be evaluated as

$$\exp\left[-g\int_0^s ds' \frac{\partial}{\partial z_\mu} A_\mu\left(\zeta(s') - 2\int_0^{s'}\Omega\right)\right].$$

It is the existence of the map [Eq. (4)] which leads to the possibility of chaotic behavior in this context of vectorial interaction. With the representation $\Omega_\mu(s') = dX_\mu/ds'$, Eq. (4) may be converted to a differential equation for $X_\mu(s')$ or to one for $x_\mu(s') = \zeta(s') - 2[X_\mu(s') - X_\mu(0)]$, whose solution will depend upon constants of integration specified (e.g.) in terms of

$$\zeta(s' = 0) = z - \frac{2\sqrt{s}}{\pi N}\sum_N{}'(P_N/N).$$

If the $A_\mu(x)$ chosen is sufficiently nonlinear, one may expect—and, as in Ref. [3], demonstrate—that chaotic behavior may occur for $\Omega_\mu(s')$, as expressed by a nonzero, positive value of a suitably averaged Lyapunov exponent. This means ultrasensitive dependence on initial conditions, here specified *via* the z and P_N variables; and even in the "semiclassical" approximation of neglecting all (P_N, Q_N) dependence, those initial conditions will be given in terms of the z_μ. If some form of numerical integration over z is contemplated, unavoidably small errors will be introduced into the initial $X_\mu(0)$ dependence, with possibly disastrous effects for the needed $\Omega_\mu(s')$ if the map [Eq. (4)] leads to chaotic behavior. [Chaotic behavior becomes even more probable if the higher frequency s' dependence of the (P_N, Q_N) is retained.] Thus, these exact vectorial interactions Green's functions of first quantization contain the possibility of sufficiently chaotic behavior, such that this exact representation generalizing Eq. (2) is in doubt. It should be noted that those few, special choices of $A_\mu(z)$ which lead to explicit solutions for $G_c[A]$—Coulomb potential, constant $F_{\mu\nu}$, a laser potential—do not produce chaotic behavior; while perturbative and even nonperturbative eikonal approximations will remove this possibility of chaos.

All of the scattering and bound states of potential theory may be described by such $G_c[A]$ for a specified $A_\mu(x)$, while the correlation functions of quantum field theory (QFTh) are obtained by Gaussian-weighted, functional integration over A_μ of products of the $G_c[A]$. We now argue that the possibility of vectorial-interaction chaos in potential theory is suppressed by the quantum field fluctuations of QFTh, wherein our previous $A_\mu(x)$ are now denoted by fixed, classical $A_\mu^{\mathrm{ext}}(x)$, plus a fluctuating component $A_\mu(x)$: $A_\mu \to A_\mu^{\mathrm{ext}} \to A_\mu^{\mathrm{ext}} + A_\mu$. In Ref. [3] we have represented the simplest, 2-point fermion correlation function $S_c'(x,y\,|\,A^{\mathrm{ext}})$ and here sketch a similar computation for this charged, scalar boson, taking all its radiative corrections into account. Neglecting, for simplicity, closed bosonic loops, one has the expression

$$\Delta_c'(x,y\,|\,A^{\mathrm{ext}}) = e^{\mathcal{D}} G_c(x,y\,|\,A^{\mathrm{ext}} + A)\,|_{A\to 0}, \qquad (5)$$

where $\mathcal{D} \equiv -(i/2)\int(\delta/\delta A_\mu)D_{c,\mu\nu}(\delta/\delta A_\nu)$ and $D_{c,\mu\nu}$ is the photon propagator.

One may now use Eq. (3) to evaluate Eq. (5) and obtain

$$\mathcal{N}'\int d[\Omega]\int d[\phi]e^{i\int_0^s f_\mu(s')\phi_\mu(s')}\exp\left[\frac{ig^2}{2}\int_0^s ds_1\int_0^s ds_2\,\phi_\mu(s_1)D_{c,\mu\nu}\left(\zeta(s_1) - \zeta(s_1) - 2\int_{s_2}^{s_1}\Omega\right)\phi_{\nu(s_2)}\right] \qquad (6)$$

with $f_\mu(s') = \Omega_\mu(s') - gA_\mu^{\mathrm{ext}}(\zeta(s') - 2\int_0^{s'}\Omega)$. The integral $\int d[\phi]$ is now Gaussian and yields, instead of a delta functional expressing the condition (4), the result

$$[\mathcal{N}']^{1/2}\int d[\Omega]\,\mathcal{F}[\Omega]e^{-(1/2)\mathrm{Tr}\ln K}\exp\left[-\frac{i}{2}\int_0^s ds_1\int_0^s ds_2\,f_\mu(s_1)\langle s_1|(K^{-1})_{\mu\nu}|s_2\rangle f_\nu(s_2)\right] \qquad (7)$$

where $\mathcal{F}[\Omega] = \exp\{-i\int_0^s ds'[p - \Omega(s')]^2\}$, and $\langle s_1|\times K_{\mu\nu}|s_2\rangle = g^2 D_{c,\mu\nu}[\zeta(s_1) - \zeta(s_1) - 2\int_{s_2}^{s_1}\Omega]$.

Equation (7) is a functional integral over a kernel defined by the radiative corrections, which cannot be evaluated in closed form, but however complicated may be the final result, it is not the chaos corresponding to the delta functional which defined the map [Eq. (4)]. That delta functional was an example of "coherence" in the sense that it is defined in terms of a product of delta functions, each of which has a standard Fourier representation as an integral, or a coherent sum of a simple phase factor over an infinite range of a relevant variable. The radiative corrections of Eq. (5) then provide a clear example of what has been termed [4] "environment-induced decoherence" as the special coherence underlying the delta functional is removed by the field fluctuations, along with the map which can lead to chaos in potential theory. If the radiative corrections are of the same size or larger than chaotic fluctuations of A_μ^{ext}, no effects of chaos should be seen in the full QFTh; if, however, chaotic effects are much larger than typical radiative corrections, one might expect "irregularities" to persist in some aspects of the full, second-quantized theory. This expectation should be tested numerically or, if possible, analytically, for there is always the possibility of the existence of an equivalent "phase change" between the two representations, which could yield completely different forms depending on whether the full, radiative corrections were or were not included.

Finally, it should be noted that the phrase "induced decoherence" used here refers to a different phenomena from that of, e.g., Ref. [5]. Our effect is strictly due to the presence of vector interactions, whereas scalar QED was considered in Ref. [5] as a semiclassical, electromagnetic field in interaction with scalar fields. The decoherence of that reference concerns a possible way in which quantum states may lose their coherence and effectively provide a basis for a classical description; ours refers, specifically, to the decoherence induced by inclusion of the full radiative corrections of QED and the subsequent removal of the map (4), which could, otherwise, easily lead to chaotic behavior of the exact correlation functions of the complete quantum field theory.

A detailed version of these remarks for the electron propagator QED, as well as various directions for further inquiry based upon these ideas, is given in Ref. [3]. It is a pleasure to acknowledge helpful conversations and correspondence with H.-T. Elze, J.-D. Fournier, U. Frisch, and W. Zurek.

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[1] E. S. Fradkin, Nucl. Phys. **76**, 588 (1966). Derivations in a variety of cases, and their infrared approximations, are given by H. M. Fried, *Functional Methods and Eikonal Models* (Editions Frontiéres, Gif-sur-Yvette, France, 1990).

[2] H. M. Fried and Y. Gabellini, Phys. Rev. D **51**, 890 (1995); **51**, 906 (1995).

[3] H. M. Fried, Y. Gabellini, and R. H. J. McKellar, Phys. Rev. D (to be published).

[4] H.-T. Elze, Nucl. Phys. **B436**, 213 (1995); CERN-TH Report No. 7997/94; W. Zurek and J. P. Paz, Los Alamos Report No. LA-UR94-927, and other references quoted therein.

[5] C. Kiefer, Phys. Rev. D **46**, 1658 (1992).