Spectral Statistics of Mesoscopic Wires: Crossover from Wigner-Dyson to Poisson Regime

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We calculate the density of states autocorrelation function $K(L,\omega)$ of a quasi-one-dimensional wire for practically all relevant values of the sample length L and the energy separation ω. As a result we obtain an overview including known types of spectral correlations for L smaller than the localization length ξ, novel spectral behavior emerging in the localized regime, $L \gg \xi$, and Poisson statistics in the thermodynamic limit. The analysis is performed within the framework of the supersymmetric nonlinear σ model.

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Energy spectra of mesoscopic metals exhibit pronounced correlations as long as the disorder is too weak to cause localization. Upon increasing the disorder strength, a transition to the insulating phase takes place and most of the system's eigenstates become uncorrelated. Eventually, one obtains Poissonian level statistics, *if* a thermodynamic limit is understood. For a long time, the strongly correlated regimes, characteristic for weakly disordered metals, and the extreme Poisson limit existed as disconnected islands in a two-dimensional parameter space spanned by disorder concentration and system size. Neither the type of correlations appearing in the crossover region nor the spectral statistics of *finite* localized systems were known.

Tremendous progress has been made in a series of recent papers [1–3] which led to the proposal of a sort of phase diagram [3] accounting for all regions intermediate between the metallic and the insulating regime. As for the metallic side of the metal insulator transition, a coherent picture was obtained by combining diagrammatic perturbation theory, renormalization group approach, and scaling arguments. Because of the lack of a truly microscopic approach, however, the analysis of the insulating phase turned out to be more problematic. To describe this side of the transition, it was *assumed* that a d-dimensional system of size L much larger than the localization length ξ decouples into $(L/\xi)^d$ uncorrelated subsystems of volume ξ^d. As a consequence, the density of states (DOS) autocorrelation function $K(L,\omega) = \nu^{-2}\langle \delta\nu(E+\omega)\,\delta\nu(E)\rangle$ [$\nu(E)$ denotes the DOS, $\langle\cdots\rangle$ the disorder average, $\nu := \langle\nu(E)\rangle$, and $\delta\nu(E) = \nu(E) - \nu$] obeys the scaling law

$$K(L,\omega) \simeq \left(\frac{\xi}{L}\right)^d f(\omega/\Delta_\xi) \xrightarrow{L\to\infty} 0, \qquad (1)$$

where Δ_ξ is the level spacing corresponding to a single localization volume and $f(\omega/\Delta_\xi) = K(\xi,\omega)$ is the associated correlation function. Qualitatively, $K(L,\omega)$ agrees with the type of correlation functions appearing in the metallic regime.

Yet, the above-mentioned picture of strictly independent ξ^d volumes is certainly oversimplified. In reality, the overlapping tails of localized wave functions induce a correlation of neighboring localization volumes, and it is by no means evident that this mechanism does not lead to a qualitatively different result. In the present Letter we show that level correlations characteristic for the localized regime *do* in fact exist. This is done by calculating the autocorrelation function $K(L,\omega)$ of a quasi-one-dimensional wire of length L for arbitrary values of L and ω. As a result, we obtain a general overview scenario including the known types of metallic spectral statistics, the extreme Poisson limit, as well as novel types of correlations emerging in the localized phase. It has to be noted that our findings for the region intermediate between metallic diffusion and localization are not directly comparable to the analyses [1–3]. Contrary to the type of systems addressed in these references, quasi-one-dimensional wires do not exhibit a true second-order metal insulator transition. Localization is inevitably observed once the system size exceeds the localization length $\xi \simeq S k_F^{d-1} l$, where k_F denotes the Fermi momentum, S the wire's cross section, and l the elastic mean free path. Notwithstanding these differences, it will become apparent below that the mechanism causing level correlations characteristic for the localized phase, $L > \xi$, applies to systems of higher dimensionality as well.

In the following we sketch the calculation of the function $K(L,\omega)$ within the framework of the supersymmetric nonlinear σ model. For the sake of computational simplicity, we consider the case of unitary symmetry, i.e., broken time reversal invariance. Following Ref. [4] we represent $K(L,\omega)$ in terms of a functional integral over a field of four-dimensional supermatrices Q,

$$K(L,\omega) = -\frac{1}{32}\,\Re \int \mathcal{D}Q e^{S[Q]} \int_0^L P^+(Q) \int_0^L P^-(Q),$$

$$S[Q] = S\int_0^L \left[\frac{\pi D\nu}{4}\,\mathrm{str}(\nabla Q)^2 + \frac{i\pi\nu\omega}{2}\,\mathrm{str}(Q\Lambda)\right], \qquad (2)$$

where $P^{\pm}(Q) = \mathrm{str}[(Q-\Lambda)(1\pm\Lambda)k]$, $\Lambda = \mathrm{diag}(1,1,-1,-1)$, $k = \mathrm{diag}(1,-1,1,-1)$, and "str" denotes the generalization of the matrix trace to graded spaces. The derivation of Eq. (2) can be found in Ref. [4]. Due to the presence of the symmetry-breaking ω-dependent

term in the action $S[Q]$, the functional integral Eq. (2) cannot be performed directly. Instead, one may apply the transfer-matrix method [4], which amounts to a mapping of the integral to a set of equivalent differential equations. To illustrate the procedure, let us consider the auxiliary quantity

$$Y_0(Q,t) = \int \mathcal{D}Q e^{S[Q,x]}\Big|_{Q(x)=Q}, \qquad t = \frac{x}{\xi}. \tag{3}$$

It can be shown [4] that Y_0 obeys the differential equation

$$\left[-\partial_t + \frac{1}{16}\Delta_Q - \frac{i}{2}\frac{\omega}{\pi\Delta_\xi}\,\mathrm{str}[Q,\Lambda]\right] Y_0(Q,t) = 0, \tag{4}$$

subject to the initial condition $Y_0(\lambda,0) = 1$. Here Δ_ξ is the mean level spacing corresponding to a sample of length $L = \xi$, and Δ_Q denotes the Laplacian acting on the manifold of Q matrices. The relation between Eq. (4) and Eq. (3) has its analog in conventional quantum mechanics: Regarding the parameter t as a sort of time variable and the Q matrices as fundamental degrees of freedom (analogous to position or spin), Eq. (4) is nothing but the time-dependent Schrödinger equation corresponding to the functional integral Eq. (3).

As it stands, Eq. (4) represents a second-order differential equation in nine variables, some of them anticommuting. A drastic simplification arises upon introducing coordinates which are tailored to the symmetries of the problem. The "potential" $V(Q) = (i\pi\nu S\omega/2)\,\mathrm{str}(Q\Lambda)$ is invariant under transformations $Q \to k^{-1}Qk$, $k \in K$, where K is the subgroup defined by $[K,\Lambda] = 0$. Proceeding in complete analogy to the treatment of spherically symmetric potentials in quantum mechanics, one may represent the Q matrices in terms of "angular" coordinates parametrizing the symmetry group K as well as "radial" coordinates $\lambda = (\lambda_1,\lambda_2)$; $\lambda_1 \in [1,\infty[$, $\lambda_2 \in [-1,1]$ accounting for the remaining degrees of freedom. As a result, Eq. (4) transforms into the purely radial equation

$$[-\partial_t + \mathcal{O}]\,Y_0(\lambda,t) = 0, \quad Y_0(\lambda,0) = 1,$$
$$\mathcal{O} = \tfrac{1}{16}\Delta_r + V(\lambda), \tag{5}$$

where $\Delta_r = 4\lambda_-^2\sum_{i=1}^2 \partial_{\lambda_i}(|1-\lambda_i|^2/\lambda_-^2)\,\partial_{\lambda_i}$ denotes the radial part of the Laplacian, $\lambda_- := \lambda_1 - \lambda_2$, and $V(\lambda) = -i(\omega/\pi\Delta_\xi)\,\lambda_-$ is the potential expressed in terms of the λ coordinates.

To complete the reformulation of Eq. (2) we define a second auxiliary quantity Y_- by

$$[-\partial_t + \mathcal{O}]\,Y_-(\lambda,t) = Y_0(\lambda,t), \quad Y_-(\lambda,0) = 0. \tag{6}$$

Expressed in terms of Y_0 and Y_-, $K(L,\omega)$ takes a form which no longer involves a functional integration:

$$K(L,\omega) = -\frac{2}{t}\,\Re\int d[\lambda]\int_0^t dt'\,Y_0(\lambda,t-t')\,\lambda_- Y_-(\lambda,t'),$$
$$\int d[\lambda] := \int_{-1}^{1} d\lambda_2 \int_1^\infty d\lambda_1\,\lambda_-^{-2}, \tag{7}$$

where henceforth $t = L/\xi$.

Equations (5), (6), and (7) are completely equivalent to the original representation Eq. (2). The solution of the differential equations depends sensitively on the value of the ratio ω/Δ_ξ. For $\omega > \Delta_\xi$ ($\omega < \Delta_\xi$), the potential term V dominates over (is small in comparison with) the "kinetic" term $\frac{1}{16}\Delta_r$. In the following, we discuss both cases separately.

$\omega > \Delta_\xi$: The strength of the potential confines λ to stay close to the origin (1,1), implying that Δ_r can be approximated by the Laplacian of a flat manifold: $\Delta_r \simeq 8\lambda_-\partial_{\lambda_-}$, in which case the differential equations can be solved analytically.

Physically, $\omega > \Delta_\xi$ represents the region of metallic diffusion. According to the semiclassical analogy between spectral correlations and classical dynamics [5], we are probing a classical particle's motion on time scales $\tau < \Delta_\xi^{-1} \simeq \xi^2/D$, which are too short to explore an entire localization volume. Thus, localization is not yet felt on these energy (time) scales. For large samples $L > \xi$,

$$K(L,\omega) \overset{L>\xi,\omega>\Delta_\xi}{\simeq} -\frac{\xi}{4L}\left(\frac{\Delta_\xi}{\omega}\right)^{3/2}, \tag{8}$$

in agreement with the results obtained for diffusive systems by Altshuler and Shklovskii (AS) [6]. Equation (8) indicates that a particle can diffuse freely for all values $\tau < \xi^2/D$ no matter how large the system is. In short systems, $L < \xi$, the prevalence of unbound diffusion is limited by the system size instead of ξ, implying that the approximation Eq. (8) applies only up to energy values $\omega \simeq E_c = D/L^2$. On smaller energy scales one is probing the fully ergodic regime which can be described by means of random matrix theory [4]. Indeed, for $\Delta_\xi < \omega < E_c$, $K(L,\omega) \simeq -1/2(\Delta/\pi\omega)^2$ which is the smooth part of the correlation function obtained from a Gaussian unitary ensemble [7].

$\omega < \Delta_\xi$: In this regime, Eqs. (5) and (6) are no longer analytically solvable. As an alternative to a direct numerical integration, one may resort to a formal eigenfunction representation of the operator $\mathcal{O}$. This strategy is motivated by the observation that the "large-t" physics is governed by a few low lying eigenvalues, which are certainly much easier to compute than the complete solution of the differential equations.

To ease the eigenfunction decomposition, we subject the operator $\mathcal{O}$ to a similarity transformation $\mathcal{O} \to \lambda_-^{-1}\mathcal{O}\lambda_-$. As a result, $\mathcal{O}$ becomes separable, i.e., $\mathcal{O} = \mathcal{O}_1 + \mathcal{O}_2$, where $\mathcal{O}_a$, $a = 1,2$, involves only the coordinate λ_a. The eigenfunctions of the transformed operator can thus be represented as product states, $|\psi_i\rangle = |\psi_{i,1}\rangle \otimes |\psi_{i,2}\rangle$, with associated eigenvalues $\epsilon_i = \epsilon_{i,1} + \epsilon_{i,2}$, where $\mathcal{O}_a|\psi_{i,a}\rangle = \epsilon_{i,a}|\psi_{i,a}\rangle$. Assuming completeness, $\sum_i |\psi_i\rangle\langle\psi_i| = 1$, and normalizability with respect to the scalar product $\langle\psi|\phi\rangle := \int[d\lambda]\,\lambda_-^2\,\psi(\lambda)\,\phi(\lambda)$, it is not difficult to show that the correlation function takes

the form [8]

$$K(L,\omega) = -\frac{1}{2}\Re\sum_{i,j,k}\langle c_i^0\psi_i|\lambda_-|\psi_j\rangle\langle\psi_j|\lambda_-|c_k^0\psi_k\rangle \times(\omega_{ij}^{-1}\omega_{jk}^{-1}e^{-\epsilon_i t} + \omega_{jk}^{-1}\omega_{ik}^{-1}e^{-\epsilon_k t} + \omega_{ij}^{-1}\omega_{ik}^{-1}e^{-\epsilon_j t}), \quad (9)$$

where $\omega_{ij} := \epsilon_i - \epsilon_j$. The final steps of the calculation, that is, the computation of eigenvalues, eigenfunctions, and the r.h.s. of Eq. (9) were performed numerically. For practically all values of (L,ω), the lowest $\mathcal{O}(10^2)$ eigenvalues were sufficient to obtain precise results.

A typical representative of $K(L,\omega)$ corresponding to a diffusive sample, $t < 1$, is shown in Fig. 1. Essentially we rediscover the findings of previous analyses, i.e., GUE-type correlations for small energies $\omega < E_c$ [4] and an overall $\sim\omega^{-3/2}$ power law decay for $\omega > E_c$ [6]. On top of the smooth $\sim\omega^{-3/2}$ background, however, we obtain a novel fine structure which oscillates with the period of the level spacing Δ. The existence of these fluctuations has recently been predicted [9] on the basis of a simple consideration: It is known (cf., e.g., Ref. [5]) that the ω-Fourier transform of $K(L,\omega)$, $K(L,\tau)$ exhibits a nonanalyticity at $\tau \simeq \Delta$. Upon Fourier transforming, the latter causes Δ-periodic fluctuations which persist throughout the *entire* spectrum. Technically speaking, these fluctuations represent a nonperturbative effect, which is why they were not resolved within earlier diagrammatic analyses [6].

Upon increasing the system size beyond the threshold $t = 1$, a slow crossover toward a universal type of spectral correlations, characteristic for the *localized* regime takes place (cf. Fig. 2). The essential features of the large-t statistics can be summarized as follows: (i) The scaling law Eq. (1) applies. (ii) The relation $f(\omega/\Delta_\xi) = K(\xi,\omega)$ applies only in the high energy regime $\omega > \Delta_\xi$, where $f(x) = x^{-3/2}$, as discussed above. (iii) For small energies $f(x) \propto \ln(x)$, $e^{-L/\xi} < x \ll 1$, in contrast to $K(\xi,\omega < \Delta_\xi) \simeq -1$. (iv) A sum rule $\lim_{\omega_c\to\infty}\int_{-\omega_c}^{\omega_c} d\omega\, K(L,\omega) = 0$ (cf., e.g., Ref. [10]) applies. Note, however, that Eq. (1) implies that the spectrum becomes increasingly more "compressible" as L grows larger (that is, the number of levels included in the interval $[-\omega_c,\omega_c]$ has to increase in order to fulfill the sum rule).

As opposed to the high energy regime $\omega > \Delta_\xi$, the behavior of the correlation function for small energies $\omega < \Delta_\xi$ is governed by localization effects. Phenomenologically, the logarithmic dependence of the scaling function f can easily be understood within a model of "correlated localization volumes" [11]. To begin with, let us consider an oversimplified model which neglects any correlation between neighboring ξ volumes. As a result, the Hamiltonian separates, $H = \sum_{i=1}^{t} H_i$, where H_i corresponds to the ith localization volume. Typically each H_i contributes a single eigenstate ψ_i with associated energy ϵ_i per energy interval Δ_ξ, and the projection of H onto an Δ_ξ-energy window takes the form $\hat{H} = \mathrm{diag}(\epsilon_1,\ldots,\epsilon_t)$ when represented in the preferential basis $\{\psi_i, i = 1,\ldots,t\}$. Upon abandoning the condition of no correlation between different volumes, the matrix $\hat{H}$ acquires off-diagonal elements $\Delta_{ij} \sim \Delta_\xi e^{\phi_{ij}} e^{-r_{ij}/\xi}$, where r_{ij} is the mean separation between volume i and j, ϕ_{ij} a random phase, and the appearance of Δ_ξ as a reference energy scale follows from normalization arguments. Although the eigenvalue statistics of the matrix $\hat{H}$ is still difficult to analyze, one may probe its relevance for the description of the real spectrum by considering the regime of asymptotically small energies. In this limit, the pairwise repulsion of (energetically) neighboring levels dominates. It is thus sufficient to analyze the statistics of the 2×2 submatrix containing those levels ϵ_i and ϵ_j which come closest to the small

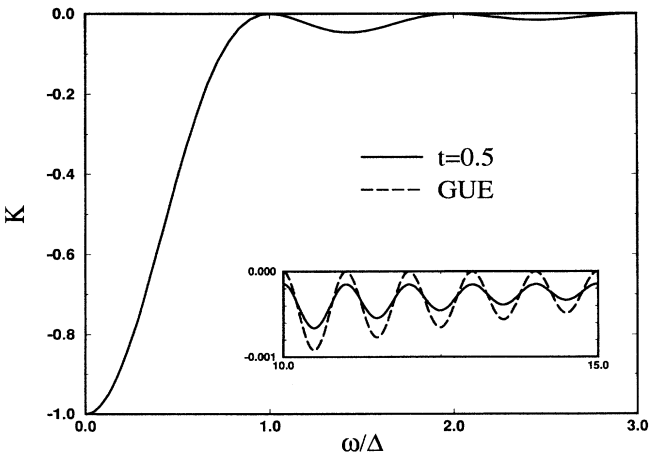

FIG. 1. Spectral correlation function of a metallic wire, $L/\xi = 0.5$.

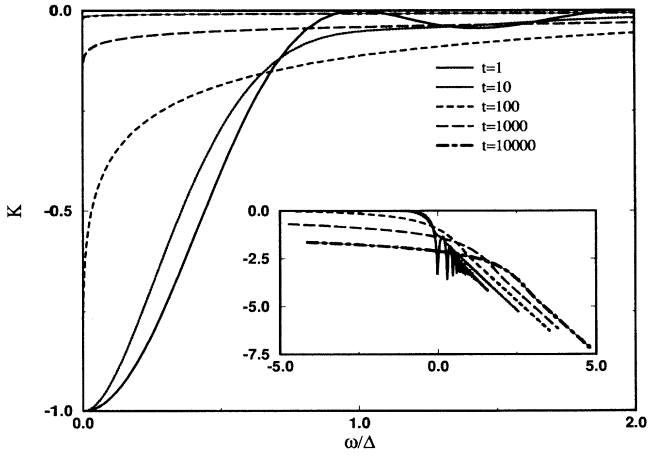

FIG. 2. Spectral correlation functions in the localized regime. $L/\xi = 1, 10, 100, 1000, 10\,000$. Inset: Double logarithmic representation.

energy window under consideration. Assuming that the indices i and j corresponding to the critical levels vary randomly between 1 and t, we are led to consider the statistics of the matrix

$$\begin{pmatrix} \epsilon & \kappa e^{-\delta r/\xi} \\ \kappa^* e^{-\delta r/\xi} & \epsilon' \end{pmatrix},$$

where ϵ and ϵ' are Gaussian distributed with width $\Delta = \Delta_\xi/t$, $|\kappa| \sim \Delta_\xi$, and $\delta r \in [0, L]$ is equally distributed. The corresponding eigenvalue correlation function can easily be computed, and upon matching numerical factors, we indeed rediscover the logarithmic law displayed above. When applied to a system of dimensionality $d > 1$, the same argument leads to the low energy behavior $K(L, \omega) \sim (\xi/L)^d [\ln(\omega/\Delta_\xi)]^d$.

Our main findings are summarized in Fig. 3. For diffusive systems $L < \xi$, we reobtain the results of previous works. As a novel feature we discover the existence of a Δ-periodic fine structure modulating the spectral correlation function in the AS regime $E_c < \omega < 1/\tau_{el}$, where τ_{el} is the elastic scattering time. This demonstrates that the spectrum "memorizes" the position

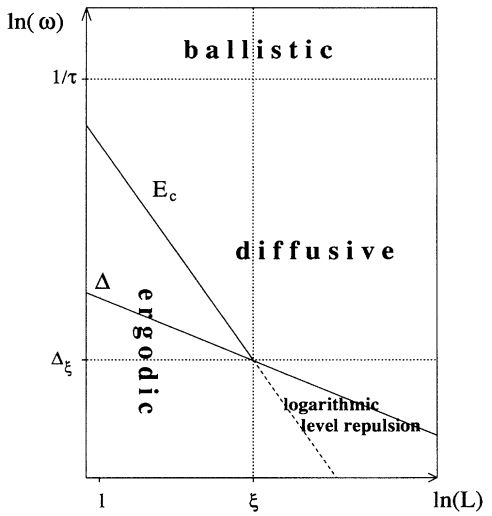

FIG. 3. Phase diagram indicating regimes with different types of spectral behavior.

of *individual* levels over large distances $\omega > E_c$. In the localized regime $L > \xi$, we obtain AS type of statistics for energies $\omega > \Delta_\xi$, yet there are no oscillatory modulations (reflecting the fact that the global level spacing Δ loses its significance). For small energies $\omega \ll \Delta_\xi$, the spectral correlation function exhibits logarithmic behavior. Its analytic dependence in the regime of intermediate energies $\Delta \ll \omega < \Delta_\xi$ is unknown to us. The scaling law Eq. (1) applies over the whole spectrum, implying a Poissonian limit $\lim_{L\to\infty} K(L, \omega) = 0$ for any fixed value of ω.

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