

Glassy Solutions of the Kardar-Parisi-Zhang Equation

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It is shown that the mode-coupling equations for the strong-coupling limit of the Kardar-Parisi-Zhang equation have a solution for $d > 4$ such that the dynamic exponent z is 2 (with possible logarithmic corrections) and that there is a delta-function term in the height correlation function $\langle h(\mathbf{k}, \omega) h^*(\mathbf{k}, \omega) \rangle = (A/k^{d+4-z})\delta(\omega/k^z)$ where the amplitude A vanishes as $d \rightarrow 4$. The delta-function term implies that some features of the growing surface $h(\mathbf{x}, t)$ will persist to all times, as in a glassy state.

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A simple nonlinear Langevin equation has been proposed by Kardar, Parisi, and Zhang (KPZ) [1] and is now widely accepted as describing the macroscopic properties of a wide variety of growth processes, such as the Eden model, growth by ballistic deposition, and the growth of an interface in a random medium [2]. This equation is also related to other seemingly disparate problems such as randomly stirred fluids [3] (Burgers equation), dissipative transport in the driven-diffusion equation [4], the directed polymer problem in a random potential [5], and the behavior of flux lines in superconductors [6]. Because of its ubiquity, any advance in understanding the KPZ equation is likely to have wide significance in both the fields of nonequilibrium dynamics and disordered systems.

The KPZ equation for a stochastically growing interface is

$$\frac{\partial h(\mathbf{x}, t)}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(\mathbf{x}, t). \quad (1)$$

It describes the large-distance, long-time dynamics of the growth process specified by a single-valued height $h(\mathbf{x}, t)$ (i.e., one with no overhangs or voids) on a d -dimensional substrate $\mathbf{x} \in R^d$. This equation reflects the competition between the surface tension smoothing forces $\nu \nabla^2 h$, the tendency for growth to occur preferentially in the direction of the local normal to the surface, represented by the nonlinear term in Eq. (1), and the Langevin noise term η which is added to model the stochastic nature of this growth process. The noise has zero mean and is Gaussian such that

$$\langle \eta(\mathbf{x}, t) \eta(\mathbf{x}', t') \rangle = 2D \delta^d(\mathbf{x} - \mathbf{x}') \delta(t - t'), \quad (2)$$

where D specifies the noise amplitude.

The objective is to characterize the form of the surface. Commonly studied are the correlation function

$$C(\mathbf{k}, \omega) = \langle h(\mathbf{k}, \omega) h^*(\mathbf{k}, \omega) \rangle \quad (3)$$

and the response function

$$G(\mathbf{k}, \omega) = \frac{1}{\delta^d(\mathbf{k} - \mathbf{k}') \delta(\omega - \omega')} \left\langle \frac{\partial h(\mathbf{k}, \omega)}{\partial \eta(\mathbf{k}', \omega')} \right\rangle. \quad (4)$$

The correlation and response functions take the following scaling forms:

$$C(\mathbf{k}, \omega) = \frac{1}{k^{2\chi+d+z}} n\left(\frac{\omega}{k^z}\right), \quad (5)$$

$$G(\mathbf{k}, \omega) = \frac{1}{k^z} g\left(\frac{\omega}{k^z}\right). \quad (6)$$

For $d > 2$, there are two distinct regimes. There is a weak-coupling regime, for $\lambda < \lambda_c$, where the nonlinear term is irrelevant and $z = 2$ and $\chi = (2 - d)/2$. For $\lambda > \lambda_c$, the nonlinear term is relevant and the scaling relation $\chi + z = 2$ follows from the invariance of Eq. (1) to an infinitesimal tilting of the surface $h \rightarrow h + \mathbf{v} \cdot \mathbf{x}$, $\mathbf{x} \rightarrow \mathbf{x} - \lambda \mathbf{v} t$ [3]. There is thus only one independent exponent to be determined in the strong-coupling regime (which is the only one we shall consider here).

Because there are no obvious small parameters to describe the strong-coupling regime, most studies of it have been numerical [7]. The most recent numerically determined values of the dynamic exponent z seem to lie between the Wolf-Kertesz [8] conjecture $\chi/z = 1/(2d + 1)$ and that of Kim and Kosterlitz [9] $\chi/z = 1/(d + 2)$ when $d < 4$. For $d \geq 5$, χ/z is still apparently nonzero but lies below the values predicted by both conjectures. Thus there is some very weak numerical evidence that the upper critical dimension d_c beyond which $z = 2$ and $\chi = 0$ is 4. In our studies we find $d_c = 4$, and we believe that for $d > 4$ the apparent nonzero values of χ arise from logarithmic factors masquerading as small powers.

We take a nonperturbative approach to the strong-coupling regime called mode-coupling theory [4]; in it one retains in the diagrammatic expansion for C and G only diagrams which do not renormalize the three-point vertex λ . This procedure leads to the following coupled equations:

$$G^{-1}(\mathbf{k}, \omega) = G_0^{-1}(\mathbf{k}, \omega) + \lambda^2 \int \frac{d\Omega}{2\pi} \int \frac{d^d q}{(2\pi)^d} [\mathbf{q} \cdot (\mathbf{k} - \mathbf{q})][\mathbf{q} \cdot \mathbf{k}] G(\mathbf{k} - \mathbf{q}, \omega - \Omega) C(\mathbf{q}, \Omega), \quad (7)$$

$$C(\mathbf{k}, \omega) = C_0(\mathbf{k}, \omega) + \frac{\lambda^2}{2} |G(\mathbf{k}, \omega)|^2 \int \frac{d\Omega}{2\pi} \int \frac{d^d q}{(2\pi)^d} [\mathbf{q} \cdot (\mathbf{k} - \mathbf{q})]^2 C(\mathbf{k} - \mathbf{q}, \omega - \Omega) C(\mathbf{q}, \Omega), \quad (8)$$

where $G_0(\mathbf{k}, \omega) = (\nu k^2 - i\omega)^{-1}$ is the bare response function and $C_0(\mathbf{k}, \omega) = 2D|G(\mathbf{k}, \omega)|^2$. Some of us [10] have recently shown that the mode-coupling equations arise from the large- N limit of a generalization of the KPZ equation to an N -component model. In principle, this might allow a systematic expansion in $1/N$, by which one could go systematically beyond mode-coupling equations towards a solution to the full problem.

In the strong-coupling limit, the scaling functions $n(x)$ and $g(x)$, where $x = \omega/k^z$, satisfy the equations

$$g^{-1}(x) = -ix + P_1 I_1(x), \quad (9)$$

$$n(x) = \frac{1}{2} P_1 |g(x)|^2 I_2(x), \quad (10)$$

where $P_1 = \lambda^2/2^d \Gamma((d-1)/2) \pi^{(d+3)/2}$. The integrals I_1 and I_2 are given by [11]

$$I_1(x) = \int_0^\pi d\theta \sin^{d-2}\theta \int_{-\infty}^{+\infty} dy \int_0^\infty dq \cos\theta(\cos\theta - q) \frac{q^{2z-3}}{r^z} g\left(\frac{x - q^z y}{r^z}\right) n(y), \quad (11)$$

$$I_2(x) = \int_0^\pi d\theta \sin^{d-2}\theta \int_{-\infty}^{+\infty} dy \int_0^\infty dq (\cos\theta - q)^2 \frac{q^{2z-3}}{r^\Delta} n\left(\frac{x - q^z y}{r^z}\right) n(y), \quad (12)$$

where $r = (1 + q^2 - 2q \cos\theta)^{1/2}$ and $\Delta = d + 4 - z$.

In the strong-coupling limit, the bare term in D in Eq. (7) can be dropped, as can the term νk^2 in the bare propagator G_0 . Equations (9)–(12) are valid for the limit $\omega \rightarrow 0$, $k^z \rightarrow 0$ with ω/k^z fixed. Notice that provided $z < 2$ no cutoff is needed.

The numerical solution of the mode-coupling equations in Eqs. (9)–(12) presents formidable problems. Recently, Tu [11] has attempted such a numerical solution, but the dependence which he obtained for z on d (first increasing from the exact value of $z = 3/2$ in $d = 1$ then decreasing at larger d) is so strange that we suspect that his solution cannot be accurate. We suspect from our own attempts at finding a direct numerical solution that problems can arise from the integrable singularities in Eqs. (11) and (12). However, if we first assume that $d_c = 4$ so that for $d > d_c$, $z = 2$ and that $n(x) = A\delta(x)$ then progress is possible. (We shall later confirm that such assumptions are consistent.)

If $z = 2$, it is now no longer possible to drop νk^2 from the bare propagator. Furthermore, $I_1(x)$ ceases to be well defined as the final momentum integral diverges logarithmically without a cutoff Λ . Thus even if $z = 2$ for $d = 4$, the scaling of ω is not likely to be simply with k^2 but with k^2 modified by some (unknown to us) power of $\ln(\Lambda/k)$. We have been unable to make any analytic progress once cutoffs are explicitly required. Instead we shall study the following problem in which the bare propagator is

$$G_0^{-1}(\mathbf{k}, \omega) = \nu(z^*) k^{z^*} - i\omega, \quad (13)$$

and we shall imagine that z^* is arbitrarily close to 2. With $z^* < 2$ no cutoff is required. Setting $n(x) = A\delta(x)$, Eq. (9) then becomes

$$g^{-1}(x) = g^{-1}(0) - ix + P_1(I_1(x) - I_1(0)). \quad (14)$$

The integrals defining the difference $I_1(x) - I_1(0)$ are convergent without a cutoff and can be calculated with $z = 2$. Also $g^{-1}(0) = \nu(z^*) + I_1(0)$ where $I_1(0) = \lambda^2 A g(0) T_1(z^*, d)$ with

$$T_1(z, d) = \frac{(3 - 2z)\Gamma((d - z)/2 + 1)B(1 - z/2, z - 1)}{(4\pi)^{d/2+1}\Gamma(d/2 + 2 - z)\Gamma((d + z)/2)}, \quad (15)$$

where $B(x, y)$ is the beta function. In addition, with $n(x) = A\delta(x)$, Eq. (10) becomes

$$A = \lambda^2 A^2 |g(0)|^2 T_2(z^*, d), \quad (16)$$

where

$$T_2(z, d) = \frac{\Gamma(d/2 + 2 - 2z)B(z, z - 1)}{(4\pi)^{d/2+1}[\Gamma(d/2 + 2 - z)]^2} \times \left[\frac{d}{2} + 2(z - 1)(z - 2) \right]. \quad (17)$$

Notice that $T_1(z, d)$ is divergent as $z \rightarrow 2$ reflecting the need for a cutoff in that limit. The solution of (16), setting $z^* = 2$ (the integrals here are convergent in that limit) is

$$A|g(0)|^2 = \frac{(d - 4)(d - 2)}{P_1 d} \times \frac{4\Gamma(d/2)}{\Gamma(1/2)\Gamma((d - 1)/2)}. \quad (18)$$

Setting $g(0) = 1$ [as can always be achieved by adjusting $\nu(z^*)$], the equation for $g(x)$ is

$$g^{-1}(x) = 1 - ix - B \int_0^1 dr r^{d-1} \left[g\left(\frac{x}{r^2}\right) - 1 \right] - B \int_1^\infty \frac{dr}{r} \left[g\left(\frac{x}{r^2}\right) - 1 \right], \quad (19)$$

where $B = 4(d - 4)(d - 2)/d^2$. This equation is readily solved for $g(x)$ numerically. Notice that as $d \rightarrow 4$, $g^{-1}(x) \rightarrow 1 - ix$ and $A \rightarrow 0$.

We have therefore found an exact solution for the mode-coupling equations for $d > 4$ when the bare propagator is as given by Eq. (13) with $z^* < 2$. The model with this bare propagator (13) is in some sense “long ranged” compared to the model with the conventional bare propagator (i.e., with $z^* = 2$). If the value of z emerging from the mode-coupling approach with the new bare propagator [Eq. (13)] had been *less* than the bare z^* , that is, if the renormalized propagator were even longer ranged, then the new model (with $z^* < 2$) and the conventional model (with $z^* = 2$) would belong to the same (strong-coupling) universality class. However, this was not found. The z of the calculation is equal to the z^* of the bare propagator. One concludes that the value of z associated with the conventional “short-range” propagator must then be greater than or equal to z^* . But as z^* can be taken arbitrarily close to 2, we conclude that the true value of z associated with the true short-range propagator must be 2, up to logarithmic factors.

One can check whether the solutions for $n(x)$ and $g(x)$ are iteratively stable as follows. By writing $n(x) = A\delta(x) + p_{n+1}(x)$ in Eq. (10), etc., one sees that

$$p_{n+1}(x) = B|g(x)|^2 \left[\int_0^1 \frac{dq}{q} p_n\left(\frac{x}{q^2}\right) + \int_1^\infty \frac{dq}{q^{d+1}} [dq^2 - (d-1)] p_n\left(\frac{x}{q^2}\right) \right] + O(p_n^2). \quad (20)$$

Under iteration we found that $p_{n+1}(x) \rightarrow \lambda_R p_n(x)$ as $n \rightarrow \infty$, with the eigenvalue $\lambda_R < 1$ (which implies stability) for $d < d^* < 4.76\dots$. In fact, there is a relation between the eigenvalue λ_R and the behavior of $p_n(x)$ as $x \rightarrow 0$; if $p_n(x) \rightarrow D/x^a$ as $x \rightarrow 0$ then direct substitution into Eq. (20) shows that

$$\lambda_R = \frac{B}{2} \left[\frac{1}{a} + \frac{d}{d/2 - 1 - a} - \frac{d-1}{d/2 - a} \right]. \quad (21)$$

λ_R has a minimum as a function of a . Within our limited numerical accuracy, λ_R determined by iteration of Eq. (20) is equal to this minimum value. When $\lambda_R > 1$, i.e., when $d > d^*$, the simple delta-function solution is unstable. We then expect that $n(x) = A\delta(x) + p(x)$, where $p(x)$ is proportional to $(d - d^*)p_\infty(x)$ and $p_\infty(x)$ is the limiting form for $p_n(x)$ as $n \rightarrow \infty$.

Thus we have a solution of the mode-coupling equations for $d > 4$, which is exact in the scaling limit and

stable. It is a “glassy” solution, in that on a Fourier transforming to (\mathbf{k}, t) variables, one sees that $C(\mathbf{k}, t)$ (in the scaling limit) is constant in time. This is rather like the original definition of Edwards and Anderson [12] of spin-glass order, i.e., the spins $S_i(t)$ have such order if $C(t) = N^{-1} \sum \langle S_i(0)S_i(t) \rangle \neq 0$ as $t \rightarrow \infty$ so that the Fourier transform of $C(t)$ has a δ function in it. However, in the present case, quenched disorder is *a priori* absent, as in a “true” glass. Hence the KPZ equation may well be another interesting model where quenched disorder is “self-generated,” as recently proposed and discussed in [13]. If this scenario is correct, our implicit assumption that the correlation and response functions are time translational invariant may not be valid, and the mode coupling may have to be recast in a two-time formulation [14].

One might wonder if the glassy behavior is attributable to the approximations made in the mode-coupling equations. While our solutions of $n(x)$ are only within the context of mode coupling, it is easy to see that non-mode-coupling diagrams for C (see Fig. 1) are such that if the δ -function ansatz is inserted for the correlator within the diagram, then each of these diagrams gives only a δ -function contribution to $n(x)$. Moreover, explicit evaluation of higher-order diagrams permits a generalization of Eq. (18) which as $d \rightarrow 4^+$ takes on the form

$$A = C_2 \left(\frac{\lambda^2 A^2 |g(0)|^2}{d-4} \right) + C_3 \left(\frac{\lambda^4 A^3 |g(0)|^4}{(d-4)^2} \right) + \dots, \quad (22)$$

where C_2 and C_3 are constants. Equation (22) implies that provided there is a nontrivial solution, it will always be such that $\lambda^2 A |g(0)|^2 \sim d - 4$. Hence we expect the upper critical dimension d_c to be 4 even beyond the mode-coupling approximation.

An approximate solution of Eqs. (7) and (8) has also suggested that $d_c \approx 3.6$ [10,15]. Previously Bouchaud and Georges [16] had argued that $d_c \geq 4$ based on a comparison with directed percolation. The existence of a finite d_c is supported by a $1/d$ expansion [17]; in addition, a prediction that d_c is 4 is contained in the functional renormalization group calculation of Halpin-Healy [18].

For dimensions $d < 4$, we do not expect to see this δ function, but precursors of glassy behavior such as very long-lived peaks in $h(\mathbf{x}, t)$ are known to exist for $d = 2$ [19]. It would be valuable to do numerical studies of the scaling limit of $C(\mathbf{k}, \omega)$ for $d > 4$ to check the existence of glassy behavior.

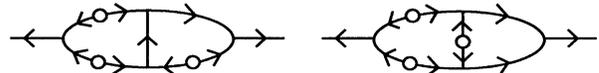


FIG. 1. Diagrams for the correlation function beyond the mode-coupling approximation. The lines with the circles within them are height correlation functions. If a δ -function form is inserted for them, the diagrams themselves give δ -function contributions to $n(x)$.

Finally, we speculate that given the solution for $d = 4$ in the mode-coupling equations it should be possible to construct a perturbative expansion for z in ϵ , where $\epsilon = 4 - d$. So far, however, we have not succeeded in this aim.

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