Short Wavelength Bifurcations and Size Instabilities in Coupled Oscillator Systems

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We report the presence of short wavelength bifurcations from synchronous chaotic states in coupled oscillator systems. The bifurcations immediately excite the shortest spatial wavelength mode present in the system as the coupling between the oscillators is increased beyond a critical value. An associated size instability places an upper bound on the number of oscillators that can support stable synchronous chaotic oscillations; an exact expression is given for the upper bound. Results are demonstrated with numerical simulations and electronic circuits.

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Coupled oscillator systems are models of choice in fields ranging from laser physics [1] to the gait of an animal's walk [2]. Basic issues in spatiotemporal behavior have been raised and decided through their study. Increasingly, the roles of chaotic behavior and its control have been the focus of coupled oscillator studies [3]. Our interest in coupled oscillators has stemmed from the attainment of synchronous chaotic behavior [4] and the desire to better understand the conditions under which this behavior can exist. In this Letter we address a related issue—that of *bifurcations* from synchronous chaotic states in coupled oscillator systems. In particular, we report what we call *short wavelength bifurcations.* These bifurcations immediately excite the *shortest* spatial wavelength mode present in the system as the coupling between the oscillators is increased beyond a critical value. This is contrary to other pattern forming bifurcations, which typically excite long or intermediate wavelengths [5]. Similar bifurcations have been reported in numerical studies of coupled map lattices [6], however, no systematic treatment of the bifurcations has been carried out. Also, there have been experimental observations of a "short wavelength instability" in an array of vortices [7]. This instability is distinct from the short wavelength bifurcation we report in two principal ways: (i) Our bifurcation emanates from a uniform chaotic state, and (ii) our bifurcation excites the shortest wavelength Fourier mode in the system. We observe short wavelength bifurcations in numerical simulations of diffusively coupled oscillators and in corresponding circuits. We also discuss an associated size instability that occurs in systems that exhibit short wavelength bifurcations. This instability limits the number of oscillators capable of sustaining stable synchronous chaos. We give a simple analytic expression for the maximum size in terms of measurable properties of the two oscillator case.

For the phenomena studied in this Letter we consider N identical diffusively coupled nonlinear oscillators with periodic boundary conditions,

$$\dot{u}_j = f(u_j) + c\Gamma(u_{j+1} + u_{j-1} - 2u_j),$$
$$j = 0, 1, \ldots, N-1, \qquad (1)$$

where $u_j \in \mathbb{R}^n$, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is nonlinear and capable of exhibiting chaotic solutions, c is a scalar coupling constant, and $\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_n)$ is a constant diagonal diffusion matrix with elements $0 \leq \gamma_i \leq 1$. By analogy with coupled map latices Eq. (1) might be called a "coupled flow lattice." Such equations could represent, for example, a discrete reaction-diffusion equation with n species [5]. Similar equations were studied by Turing [8] in an effort to explain symmetry breaking in the development of living organisms.

We are interested in bifurcations from synchronous chaotic states; these states reside on a *synchronization manifold* defined by $\mathcal{M} = \{u_0 = u_1 = \cdots = u_{N-1} = s(t)\}$, where the chaotic solution $s(t)$ satisfies the isolated oscillator equation $\dot{s} = f(s)$. The synchronization manifold $\mathcal{M}$ has the dimension of a single oscillator (n) and is invariant under the flow (1). Stability of the synchronous state can be determined by letting $u_j = s + \xi_j$ and linearizing (1) about $s(t)$. This leads to

$$\dot{\xi}_j = Df(s)\xi_j + c\Gamma(\xi_{j+1} + \xi_{j-1} - 2\xi_j), \qquad (2)$$

where $Df(s)$ is the Jacobian of f on $s(t)$. Linear stability equations such as (2) can be diagonalized by expanding into spatial Fourier modes, $\xi_j = (1/\sqrt{N})\sum_{k=0}^{N-1} \eta_k e^{-2\pi ijk/N}$. Carrying this out gives

$$\dot{\eta}_k = [Df(s) - 4c\sin^2(\pi k/N)\Gamma]\eta_k,$$
$$k = 0, 1, \ldots, N-1. \qquad (3)$$

This result is also derived in [9]. It is shown in [10] that diagonalization via Fourier mode expansion is possible for any coupled oscillator system whose coupling configuration is *shift invariant*—a requirement met by (1).

Because the ξ_j are real and $\mathrm{Re}(\eta_k)$ and $\mathrm{Im}(\eta_k)$ satisfy the same equation, only the $k = 0, 1, \ldots, N/2$ equations need to be considered to decide the stability of the synchronous state (here and throughout we assume N is even). The highest wave number (shortest wavelength) mode occurs at $k_{\max} = N/2$. The $k = 0$ mode governs motion *on* the synchronization manifold. This mode has Lyapunov exponents $\lambda_1^0 \geq \lambda_2^0 \geq \cdots \geq \lambda_n^0$, which are also those of a single oscillator exhibiting the solution $s(t)$. The synchronized state is chaotic if $\lambda_1^0 > 0$. The $k > 0$

modes represent variations *transverse* to $\mathcal{M}$. We define the *transverse Lyapunov exponents* (TLEs) [11] for the kth mode in a similar manner, $\lambda_1^k \geq \lambda_2^k \geq \cdots \geq \lambda_n^k$. A necessary condition for the stability of the synchronous state is $\lambda_1^k < 0$ for all $k > 0$. In general, there is no simple relation between the Lyapunov exponents on $\mathcal{M}$ and those transverse to $\mathcal{M}$. A notable exception is the case when Γ is the identity (vector coupling). In this case [9] $\lambda_j^k = \lambda_j^0 - 4c\sin^2(\pi k/N)$, which shows that synchronous chaos is always possible for large enough coupling c. One of the main points of this Letter is that, contrary to expectation, large coupling can *destabilize* synchronous chaotic states when the diffusion matrix Γ is other than the identity.

The structure of Eq. (3) enforces relations between the TLEs for varying mode number k and varying number of oscillators N. In particular, for a given diffusion matrix Γ all of the TLEs can be obtained from the $k = 1$ mode of the $N = 2$ oscillator case [12]. This can be seen as follows. Let the TLEs for Eq. (3) be denoted by $\lambda_j^k(c,N)$. Equation (3) can also be written as $\dot{\eta}_k = [Df(s) - 4\tilde{c}\sin^2(\pi/2)\Gamma]\eta_k$, which is recognized as the linear stability equation governing the $k = 1$ mode for $N = 2$, with coupling coefficient $\tilde{c} = c\sin^2(\pi k/N)$. Therefore the TLEs satisfy the scaling relation

$$\lambda_j^k(c,N) = \lambda_j^1(c\sin^2(\pi k/N),2). \tag{4}$$

Generally we are interested only in the largest TLE for each mode, $\lambda_1^k(c,N)$, which can be obtained from $\lambda_1^1(c,2)$ by scaling the coupling constant c according to (4). In this way the stability of the synchronized state for any size array can be determined by computing the *single* curve $\lambda_1^1(c,2)$. This curve is generated most easily by integrating (3) with $k = 1$ and $N = 2$ and computing the asymptotic growth rate of a randomly chosen initial vector $\eta_1(0)$ for each value of c.

For our numerical and experimental studies we let $u = (x,y,z)$ and focus on a variant of the Rössler system [13], defined by

$$\begin{aligned} \dot{x} &= -(\alpha x + \beta y + z), \\ \dot{y} &= x + \delta y, \\ \dot{z} &= g(x) - z, \end{aligned} \tag{5}$$

where $g(x) = 0$ for $x \leq 3$ and $g(x) = \mu x$ for $x > 3$. This system was studied in [10] and was independently introduced in [14]. The piecewise linear term in the z equation allows for easy circuit implementation (cf., [10] for circuit diagram). Equation (5) has a chaotic attractor, both numerically and experimentally, for parameters $\alpha = 0.05$, $\beta = 0.5$, $\delta = 0.133$, and $\mu = 15.0$. Figure 1 shows the largest TLE $\lambda_1^1(c,2)$ determined numerically from (3) for x,y,z and vector coupling [$\Gamma = \mathrm{diag}(1,0,0)$, $\mathrm{diag}(0,1,0)$, $\mathrm{diag}(0,0,1)$, and $\mathrm{diag}(1,1,1)$, respectively]. Only the vector coupling case is monotonic with c. The z-coupling case never synchronizes, while the x- and y-coupling cases synchronize for $c > c_1 = 0.0315$ and $c > 0.026$, respectively (cf., inset). Similar figures have been obtained by Pyragas [15] within the context of continuous feedback control. Experiments carried out with $N = 4$ coupled electronic circuits [10] exhibit synchronization thresholds in good agreement with the predictions shown in Fig. 1.

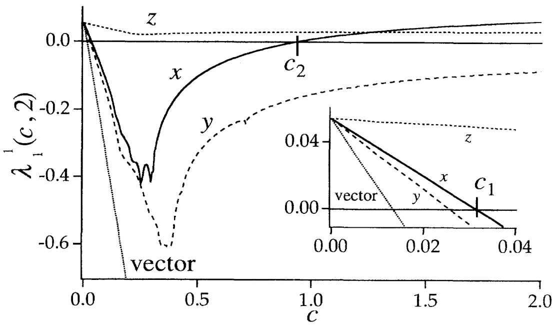

FIG. 1. Largest TLE versus coupling for first mode of coupled modified Rössler oscillators ($N = 2$). Shown are curves for x,y,z and vector coupling. Inset shows linear region for small c. Coupling constants c_1 and c_2 are zero crossings of the x-coupling curve.

In this Letter we are primarily interested in the x-coupling case, which is seen to *desynchronize* for $c > c_2 = 0.945$. Figure 2 shows the largest TLEs of mode 1 and mode 2 for $N = 4$ oscillators; these curves are found from $\lambda_1^1(c,2)$ by scaling c according to (4). Note the $\lambda_1^2(c,4) = \lambda_1^1(c,2)$. This is a general feature of the highest mode ($k = N/2$) curve; $\lambda_1^{N/2}(c,N) = \lambda_1^1(c,2)$. As c increases, mode 2 destabilizes *first* at $c = 0.945$, followed by mode 1 at twice this value, $c = 1.89$. We call the event of passing through $c = c_2$ from below a *short wavelength bifurcation* (SWB). This bifurcation immediately excites the smallest spatial scale in the system—there is no cascading of energy from longer wavelength scales. As shown below, the short wavelength bifurcation can occur only when the array size N is below a critical value.

Both numerical integrations and experiments with the coupled modified Rössler system exhibit SWBs. Figure 3 shows the numerical and experimental power (P) in the transverse modes as a function of time for $N = 4$ and x

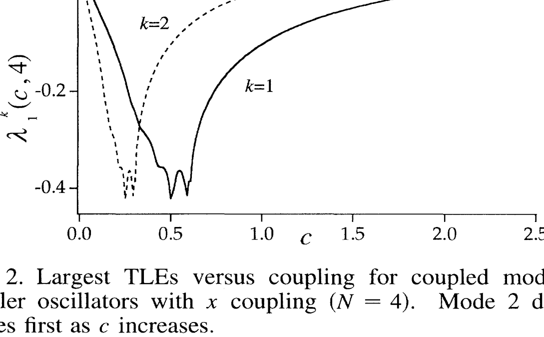

FIG. 2. Largest TLEs versus coupling for coupled modified Rössler oscillators with x coupling ($N = 4$). Mode 2 destabilizes first as c increases.

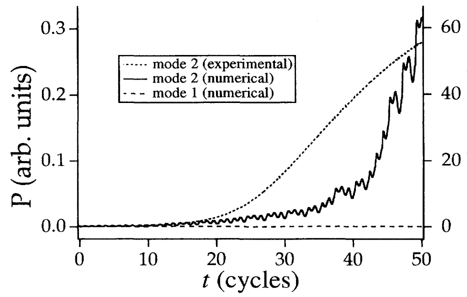

FIG. 3. Power in y component of transverse modes versus time: (left axis) numerical simulation showing growth of mode 2 following increase in c at $t = 50$ cycles; (right axis) experimental plot showing similar growth in mode 2 following change in c at $t = 0$.

coupling, immediately following a change in the coupling from just below to just above the SWB. The left axis corresponds to the numerical calculation, while the right axis corresponds to the experiment. The power was computed from the y signals of the oscillators; x and z signals give similar results. The time is in "cycles" around an isolated attractor (1 cycle $\equiv 2\pi/\sqrt{\beta}$). Mode 2 is seen to destabilize in both cases, while mode 1 remains near zero. (Only the numerical mode 1 is plotted for clarity; the experimental curve is indistinguishable from this curve at the scale of the figure.) The experiment is seen to lose stability faster than the numerical calculation. This is presumably due to larger fluctuations in the experiment or imprecise tuning between the model and the circuits.

Figure 4 shows the numerical attractors before and after the change in the coupling constant. After the change to $c = 1.0$ trajectories converge to one of *two* nonsynchronous *periodic* attractors [identified with solid and dashed lines Fig. 4(b)]. For a given attractor the periodic motion of oscillator i is the same as that of oscillator $i + 2$ mod4. Since the array is shift invariant, the two attractors are related by a simple shift of indices $i \rightarrow i + 1$ mod4. The spatial structure of each attractor is $+ - + -$, which matches the highest Fourier mode in the four oscillator system [16]. The periodic attractors are present simultaneously with the synchronous attractor, hence the SWB is similar to a subcritical pitchfork bifurcation. In other systems, namely, the "standard" Rössler system [13] with x coupling and the Lorenz system [17] with z coupling, we have found supercritical SWBs.

The behavior of the TLE curves depends on the size of the oscillator array. Systems that exhibit short wavelength bifurcations, when scaled to large enough arrays N, display a size instability that precludes the existence of stable synchronous chaos for *any* value of c. Figure 5 shows a few of the TLEs for the $N = 16$ case [again obtained from the scaling relation (4)]. The first zero crossing of the lowest mode ($k = 1$) occurs just before the second zero crossing of the highest mode ($k = 8$). There

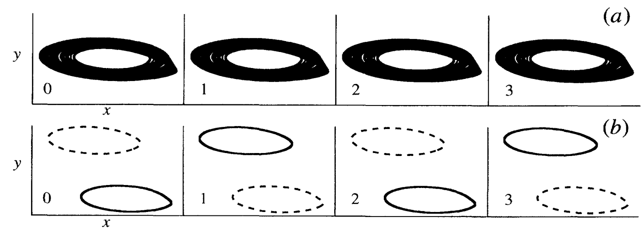

FIG. 4. Numerical attractors for x coupling and $N = 4$: (a) $c = 0.9$, synchronous chaos; and (b) $c = 1.0$, two periodic attractors, one solid and the other dashed, with short wavelength spatial variation.

is a very small region between the two zero crossings (imperceptible in the figure) where synchronous chaos is possible. For $N > 16$ this region disappears, as the zero crossings pass through one another. From the scaling relation (4) one can compute the array size, $N_{\max}$, for which the two zero crossings coincide. This yields the largest array capable of sustaining stable synchronous chaos. We wish to solve $\lambda_1^1(c,N) = \lambda_1^{N/2}(c,N) = 0$ for N. Since $\lambda_1^{N/2}(c,N) = \lambda_1^1(c,2)$, the zero crossing occurs at $c = c_2$ (cf., Fig. 1). Using the scaling relation (4) one then has $\lambda_1^1(c_2,N) = \lambda_1^1(c_2 \sin^2(\pi/N), 2) = 0$. It follows that $c_1 = c_2 \sin^2(\pi/N)$, or

$$N_{\max} = \left[\pi/\sin^{-1}\left(\sqrt{c_1/c_2}\right)\right], \qquad (6)$$

where $[\cdots]$ denotes integer part. For the x coupling case above one finds $N_{\max} = 17$, corroborating Fig. 5. Relation (6) holds for any diffusively coupled system whose basic stability curve $\lambda_1^1(c,2)$ is qualitatively similar to the x coupling curve in Fig. 1. $N_{\max}$ is determined uniquely by the two zero crossings c_1 and c_2, which are easily calculated (or measured) for the two oscillator system. Note that the zero crossings are system dependent. In principle the larger zero crossing c_2 can be arbitrarily large, yielding an arbitrarily large value of $N_{\max}$. Therefore our results are not necessarily limited to small arrays.

A necessary and sufficient condition for the SWB and associated size instability is the existence of a positive going zero crossing in the basic curve $\lambda_1^1(c,2)$ (e.g., c_2

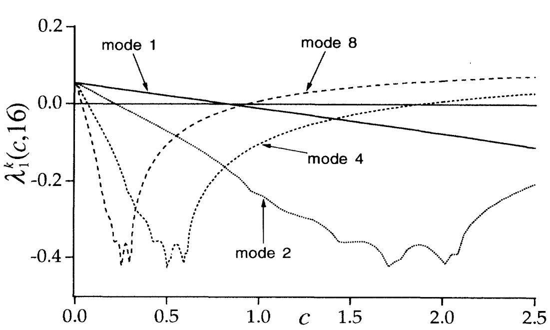

FIG. 5. TLEs for x-coupled modified Rössler system with $N = 16$. Zero crossings of $k = 1$ and $k = 8$ modes are approximately the same, showing impending size instability for $N > 16$.

in Fig. 1). It is desirable to know when this can occur, without having to integrate Eqs. (3). Here we outline a method to determine the *sign* of $\lambda_1^{\perp}(c,2)$ as c becomes large. The result can be used to support or rule out candidates for the SWB. While $\lambda_1^{\perp}(c,2) > 0$ for large c does not guarantee the SWB, since $\lambda_1^{\perp}(c,2)$ may *always* be positive (e.g., z coupling in Fig. 1), $\lambda_1^{\perp}(c,2) < 0$ for large c does rule out the SWB [18]. First, redefine the time in the $k = 1$, $N = 2$ variational equation (3) $\tau = 4ct$. This gives $d\eta/d\tau = [-\Gamma + \varepsilon Df(s)]\eta$, where $\varepsilon = 1/(4c)$. The idea is to compute the largest Lyapunov exponent of this equation via perturbation theory. The details of the calculation are presented in [10]. Here, due to space limitations, we simply state the result. Define the average sub-Jacobian on $s(t)$, $\langle J_P \rangle = P\langle Df \rangle P$ (for x coupling P is a projection matrix onto the y-z subspace, and similarity for other coupling choices). Then the largest Lyapunov exponent of the $k = 1$, $N = 2$ stability equation is, to first order in ε, $\lambda_1^{\perp}(c,2) = \varepsilon\,\mathrm{Re}(\rho_{\max})$, where $\rho_{\max}$ is the eigenvalue of $\langle J_P \rangle$ with the largest real part. The condition for stable sychronous chaos is simply $\rho_{\max} < 0$. This result correctly predicts the sign of the x-, y-, and z-coupling cases [10]. The stability condition is similar to the condition for stable synchronous chaos in one-way driving configurations [4], namely, that the *conditional Lyapunov exponents* are negative.

In summary, we have described a bifurcation from synchronous chaos in spatially extended systems that is capable of generating structure at the scale of the resolution of the spatial lattice. This short wavelength bifurcation is observable in simulations and coupled analog circuits. SWBs could play a role in strongly coupled pattern forming systems. In addition, control of behavior near SWBs may be problematic, possibly requiring a large number of control sites along the spatial lattice. Whenever a diffusively coupled system exhibits a short wavelength bifurcation, one can expect a size instability to occur as the array size increases beyond a critical value. This critical size can be predicted from measured properties of a two oscillator array.

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[1] R. Roy and K.S. Thornburg, Jr., Phys. Rev. Lett. **72**, 2009 (1994).

[2] J.J. Collins and I.N. Stewart, J. Nonlinear Sci. **3**, 349 (1993).

[3] H. Gang and Q. Zhilin, Phys. Rev. Lett. **72**, 68 (1994); D. Auerbach, *ibid.* **72**, 1184 (1994); I. Aranson, H. Levine, and L. Tsimring, *ibid.* **72**, 2561 (1994).

[4] L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. **64**, 821 (1990); Phys. Rev. A **44**, 2374 (1991).

[5] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer-Verlag, New York, 1984).

[6] L.A. Bunimovich, A. Lambert, and R. Lima, J. Stat. Phys. **61**, 253 (1990).

[7] O. Cardoso, H. Willaime, and P. Tabeling, Phys. Rev. Lett. **65**, 1869 (1990).

[8] A. Turing, Philos. Trans. R. Soc. London B **237**, 37 (1952).

[9] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983).

[10] J.F. Heagy, T.L. Carroll, and L.M. Pecora, Phys. Rev. E **50**, 1874 (1994).

[11] J.C. Alexander, I. Kan, J.A. Yorke, and Z. You, Int. J. Bifurcations Chaos **2**, 795 (1992); E. Ott, J.C. Sommerer, J.C. Alexander, I. Kan, and J.A. Yorke, Phys. Rev. Lett. **71**, 4134 (1993).

[12] For $N = 2$ the appropriate coupling term in (1) is $2c\Gamma(u_{j+1} - u_j)$.

[13] O.E. Rössler, Phys. Lett. **57A**, 397 (1976).

[14] O.E. Rössler, J.L. Hudson, and R. Rössler, Physica (Amsterdam) **62D**, 80 (1993).

[15] K. Pyragas, Phys. Lett. A **181**, 203 (1993).

[16] The *asymptotic* attractors after the bifurcation need not show the spatial structure of the shortest wavelength mode, however, the initial spatial structure always will.

[17] E.N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).

[18] This assumes that there are no positive going zero crossings for *intermediate* values of c. This is the case for the systems we have studied.