Density of Levels in a Generalized Matrix Ensemble

M. Kreynin and B. Shapiro
Department of Physics, Technion–Israel Institute of Technology, Haifa 32000, Israel
(Received 11 October 1994)

We present analytical results for the average density of levels $\rho(\lambda)$ in an ensemble of random matrices, which is a generalization of the standard Gaussian unitary ensemble. The generalized ensemble contains a parameter $\mu(N)$ which is allowed to scale, in an arbitrary way, with the matrix size N. The results crucially depend on the behavior of $\mu(N)$. For a sublinear dependence of μ on N a modified Wigner semicircle is obtained, in the large N limit. For a superlinear dependence the ensemble approaches the Poissonian limit of uncorrelated levels, with a Gaussian shape for $\rho(\lambda)$. For a strictly linear dependence, i.e., when $\lim_{N\to\infty}(\mu/N) = \text{const}$, an intermediate situation occurs.

PACS numbers: 05.40.+j, 05.45.+b

Random matrix theory [1] operates with ensembles of matrices which obey certain symmetries and constraints but otherwise are "as random as possible." For instance, the Gaussian unitary ensemble (GUE) is defined as an ensemble of Hermitian matrices H with the single constraint $\langle \mathrm{Tr}H^2\rangle_0 = \text{const}$, where $\langle\cdots\rangle_0$ denotes averaging over the ensemble. The corresponding probability distribution, in the matrix space, is

$$P_0(H)d^{N^2}H \sim e^{-\mathrm{Tr}H^2}d^{N^2}H\,, \tag{1}$$

where the volume element can be written explicitly as

$$d^{N^2}H = \prod_{i=1}^{N} dH_{ii} \prod_{i<j} d(\mathrm{Re}H_{ij})d(\mathrm{Im}H_{ij})\,.$$

[An $N \times N$ Hermitian matrix can be parametrized by its N (real) diagonal elements and $\frac{1}{2}N(N-1)$ (complex) elements above the main diagonal.] Denoting the eigenvalues of a matrix H by λ_i $(i = 1,\ldots,N)$, one defines the average density of levels as $\rho_0(\lambda) = \langle \sum_i \delta(\lambda - \lambda_i)\rangle_0$. In the large N limit, this function is given by the famous Wigner semicircle [1]:

$$\rho_0(\lambda) = \frac{1}{\pi}\sqrt{2N - \lambda^2}\,. \tag{2}$$

Recently there has been some interest in more general matrix ensembles, such as banded matrices [2,3], sparse matrices [4,5], and others [6–12]. Such generalized ensembles can interpolate between the GUE with its characteristic Wigner-Dyson statistics and a Poissonian ensemble which describes uncorrelated levels. A transition between the two statistics (i.e., Wigner-Dyson and Poisson) often occurs in disordered [13–15] and chaotic [16] systems.

The crossover between Wigner-Dyson and Poisson statistics resembles weakening of the repulsive correlation in a Fermi gas under increase of temperature. This analogy becomes precise for the generalized ensemble introduced and studied by Moshe, Neuberger, and Shapiro [11]. Another interesting analogy is suggested by the recent work on statistics of "quons" (see Ref. [17], and references cited therein). These are particles which obey the commutation relations $a_k a_l^\dagger - q a_k^\dagger a_l = \delta_{kl}$. When the parameter q increases, starting from $q = -1$ (fermions), the repulsive correlations in the particle density diminish, similar to the decrease of the level repulsion in generalized ensembles. Thus, generalized ensembles of random matrices appear in various fields of physics, including two-dimensional quantum gravity [18] and topological field theory [19].

An important feature of the GUE (and other standard Gaussian ensembles) is its invariance under unitary transformation, i.e., under change of basis. On the other hand, in realistic models of complex quantum systems this U invariance is broken. For instance, the Anderson model [20] of a disordered electronic system is naturally defined in the (real space) site representation, and the same is true for many other models in solid state physics. The existence of preferential basis in physical systems leads to deviations from predictions of the standard random matrix theory and, thus, motivates studies of matrix ensembles with broken unitary or orthogonal invariance. One possible way to break this invariance is to introduce the following generalized ensemble:

$$P(\{H_{ij}\})d^{N^2}H = \text{const} \times \exp\left\{-\sum_{i=1}^{N} H_{ii}^2 - 2(1+\mu) \times \left[\sum_{i<j}\left[(\mathrm{Re}H_{ij})^2 + (\mathrm{Im}H_{ij})^2\right]\right]\right\} d^{N^2}H\,. \tag{3}$$

The parameter μ imposes a preferential basis, namely, the basis which is used in Eq. (3). More weight is given to matrices which are nearly diagonal in that basis. When the parameter μ changes from 0 to ∞, for a fixed value of N, the ensemble changes from GUE [Eq. (1)] to the Poissonian ensemble of diagonal matrices. The ensemble defined in Eq. (3) (more precisely, its orthogonal counterpart) was studied numerically by Rosenzweig and Porter [21]. The two-point level correlation function near the Poissonian limit was computed by Leyvraz and Seligman [22]. More recently, Pichard and Shapiro [10] gave a qualitative picture of the level statistics evolution under

 0031-9007/95/74(21)/4122(3)$06.00

change of μ, and used this ensemble as a paradigm for a maximal entropy ensemble with a preferential basis.

In this Letter we initiate an analytical study of this generalized ensemble, in the $N \to \infty$ limit. Such a study may prove useful for explaining deviations from the Wigner-Dyson statistics, or from the Porter-Thomas statistics for eigenvectors [1], in disordered or chaotic quantum systems. The results presented below are limited to the average level density $\rho(\lambda)$. It is clear that a radical change from the GUE behavior, in the large N limit, can be obtained only if the parameter μ increases with N sufficiently fast. Indeed, the N diagonal elements can compete with the (roughly) N^2 off-diagonal ones only if the magnitude of the off-diagonal elements is suppressed as some power of N. It turns out that, similar to some other matrix ensembles [10,11], there are three distinct regimes.

(1) The dependence of μ on N is slower than linear, i.e., the parameter $C_N \equiv \mu/N$ approaches zero when $N \to \infty$. In this case the limiting shape of the average density of levels $\rho(\lambda)$ is

$$\rho(\lambda) = \frac{1+\mu}{\pi}\left[2\frac{N}{1+\mu} - \lambda^2\right]^{1/2}, \tag{4}$$

which is a modified Wigner semicircle. We emphasize that μ in Eq. (3) can depend on N. For instance, if $\mu = N^\alpha$ (with $0 < \alpha < 1$), Eq. (3) simplifies to $\rho(\lambda) = \pi^{-1}N^\alpha\sqrt{2N^{1-\alpha} - \lambda^2}$.

(2) μ increase with N faster than linearly, i.e., $C_N \to \infty$. In this case one ends up with an ensemble of diagonal matrices, with normally distributed elements, so that

$$\rho(\lambda) = \frac{N}{\sqrt{\pi}}\, e^{-\lambda^2}. \tag{5}$$

(3) μ grows linearly with N, i.e., $C_N \to C_0 = \text{const}$. The situation resembles the critical point of the Anderson transition, where in the thermodynamic limit a new universal statistics sets in [23,24]. In this case $\rho(\lambda) = NF(\lambda)$, where the function $F(\lambda)$ contains C_0 but does not depend on N. For an arbitrary C_0 it does not seem possible to obtain a closed analytical expression for $F(\lambda)$. When C_0 is large, $F(\lambda)$ is close to the Gaussian of Eq. (5). When C_0 is small $F(\lambda)$ is close to $\pi^{-1}\sqrt{2C_0 - (\lambda C_0)^2}$ (as long as $\lambda^2 \ll C_0^{-1}$) but develops Gaussian tails for large $|\lambda|$. For $C_0 \simeq 1$ this function can be computed numerically: Fig. 1 depicts $F(\lambda)$ for $C_0 = 1$.

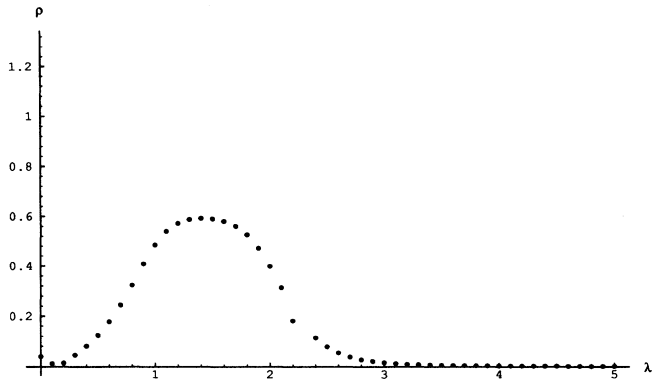

FIG. 1. The average density of levels (normalized to one level), for $C_0 = 1$. Only the positive values of λ are shown in the figure. The value of $\rho(\lambda)$ for negative values of λ is determined by the condition $\rho(-\lambda) = \rho(\lambda)$.

After having presented our main results, in the rest of the paper we outline their derivation. It is based on a generalization of the "fermionic method," presented for the GUE in Ref. [25]. The resolvent of a Hermitian matrix H is written as

$$\left(\frac{1}{z-H}\right)_{ij} \equiv G_{ij}(z) = -i\int\left(\prod_{1\le\kappa\le N}\frac{d\phi_\kappa\, d\phi_\kappa^*\, d\chi_\kappa\, d\chi_\kappa^*}{\pi}\right)\phi_i\phi_j^*\exp\left\{i\sum_{m,n=1}^{N}\left[\phi_m^*(z-H)_{mn}\phi_n + \chi_m^*(z-H)_{mn}\chi_n\right]\right\}, \tag{6}$$

where ϕ_κ and χ_κ ($\kappa = 1,\ldots,N$) denote a set of complex and Grassmann variables, respectively. After averaging G_{ij} over the distribution $P(H)$ of Eq. (3), which amounts to performing simple Gaussian integrals, and then integrating out the Grassmann variables, we obtain

$$\langle G_{ij}(z)\rangle = -i\sqrt{\frac{1+\mu}{\pi}}\int_{-\infty}^{+\infty} d\alpha\, e^{-(1+\mu)\alpha^2}\int\left(\prod_{1\le\kappa<N}\frac{d\phi_\kappa d\phi_\kappa^*}{\pi}\right) \times \phi_i\phi_j^*(\det M)\exp\left[\sum_{n=1}^{N}\left(iz\phi_n^*\phi_n - \frac{1}{4}\phi_n^{*2}\phi_n^2\right) - \frac{1}{4(1+\mu)}\sum_{n,m}\phi_n^*\phi_n\phi_m^*\phi_m\right], \tag{7}$$

where the matrix M is given by

$$M_{ij} = \left(\alpha - iz + \frac{1}{2}\phi_i^*\phi_i\right)\delta_{ij} + \frac{1}{2(1+\mu)}\phi_i\phi_j^*(1-\delta_{ij}). \tag{8}$$

For $\mu = 0$, the determinant of M was calculated in Ref. [25]. Fortunately, it can be calculated for any μ, with the following result:

$$\det M = \left[\prod_{j=1}^{N}\left(\alpha - iz + \frac{\mu}{2(1+\mu)}\phi_j^*\phi_j\right)\right]\left[1 + \frac{1}{2(1+\mu)}\sum_j \phi_j^*\phi_j\left(\alpha - iz + \frac{\mu}{2(1+\mu)}\phi_j^*\phi_j\right)^{-1}\right]. \tag{9}$$

The next stage of the calculation involves doing integrals over the complex field. The details of this rather cumbersome calculation will be presented in a separate publication. Here we only give the final result for $\langle \mathrm{Tr}G \rangle$ which is all one needs for computing the average density of levels:

$$\langle \mathrm{Tr}G(z) \rangle = -i \frac{1+\mu}{\pi} N \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\alpha \, d\gamma \, e^{-(1+\mu)(\alpha^2+\gamma^2)} \times \left[(\alpha - iz) f + \frac{\mu}{2i(1+\mu)} f' \right]^{N-2} \times \left\{ \left[(\alpha - iz) f + \frac{\mu}{2i(1+\mu)} f' \right] \left[i(\alpha - iz) f' - \frac{1}{2} f'' \right] + \frac{N-1}{2i(1+\mu)} f' \left[i(\alpha - iz) f' - \frac{\mu}{2(1+\mu)} f'' \right] \right\}, \quad (10)$$

where f denotes the function

$$f(u) = \sqrt{\pi \frac{1+\mu}{\mu}} \exp\left(-\frac{1+\mu}{\mu} u^2 \right) \times \left[1 - \mathrm{erf}\left(-iu \sqrt{\frac{1+\mu}{\mu}} \right) \right] \quad (11)$$

taken at the value $u = z - \gamma$ of its argument. Correspondingly, f' and f'' denote the first and second derivatives of f at that value of its argument.

The average density of levels, $\rho(\lambda) = -\pi^{-1} \times \mathrm{Im}\langle \mathrm{Tr}G(z = \lambda + i0) \rangle$, and the last step of the calculation requires doing the double integral in Eq. (10). For large N this is done by the method of steepest descent. Although the saddle point values α_s, γ_s cannot be found analytically in the general case, a rather complete analytical treatment is possible. This is due to the property

$$f' = 2 \frac{1+\mu}{\mu} (i - uf) \quad (12)$$

which follows directly from the expression (11) for the function f. This property ensures the relation $\alpha_s = i\gamma_s$ and highly facilitates the subsequent treatment. The saddle point equations contain the parameter $C_N \equiv \mu/N$. If C_N decreases with N, there is a pair of saddle points, as in the GUE case. The only difference from the GUE is the appearance of the factor $1 + \mu$, which results in Eq. (4) for $\rho(\lambda)$. In the opposite limit, i.e., when $C_N \to \infty$ with N, there is only one saddle point and $\rho(\lambda)$ is given by Eq. (5). For the intermediate case ($C_N \to C_0$) the saddle point equations cannot be solved analytically, for an arbitrary value of C_0. An approximate analytical treatment is possible for large or small C_0, with the results stated above. In $C_0 \simeq 1$ one has to resort to numerical computations (Fig. 1).

In conclusion, we have calculated the average density of levels $\rho(\lambda)$ for the ensemble defined in Eq. (3), in the large N limit. The functional form of $\rho(\lambda)$ crucially depends on how fast the parameter μ increases with N. There are three distinct regimes, corresponding to sublinear, superlinear, or strictly linear dependence of μ on N. It is interesting to note that the behavior of $\rho(\lambda)$ is rather similar to that obtained in Ref. [11] for a quite different matrix ensemble. This similarity suggests that different one-parameter ensembles, interpolating between the Wigner-Dyson and the Poisson statistics, may exhibit a large degree of universality.

We acknowledge useful discussions with R. Pnini and D. Schmeltzer. The research was supported by the fund for the promotion of research at the Technion and the Venezuela Technion Society Research Fund.

[1] M. L. Mehta, *Random Matrices* (Academic Press, New York, 1991), 2nd ed.
[2] A. D. Mirlin and Y. V. Fyodorov, J. Phys. A **26**, L551 (1993).
[3] M. Feingold, A. Gioletta, F. M. Izrailev, and L. Molinari, Phys. Rev. Lett. **70**, 2936 (1993).
[4] S. N. Evangelou and E. N. Economou, Phys. Rev. Lett. **68**, 361 (1992).
[5] Y. V. Fyodorov and A. D. Mirlin, Phys. Rev. Lett. **67**, 2049 (1991).
[6] T. H. Seligman, J. J. M. Verbaarschot, and M. R. Zirnbauer, Phys. Rev. Lett. **53**, 215 (1984).
[7] M. R. Berry and M. Robnik, J. Phys. A **17**, 2413 (1984).
[8] P. Cizeau and J. P. Bouchaud, Phys. Rev. E **50**, 1810 (1994).
[9] K. A. Muttalib, Y. Chen, M. E. H. Ismail, and V. N. Nicopoulos, Phys. Rev. Lett. **71**, 471 (1993).
[10] J.-L. Pichard and B. Shapiro, J. Phys. **4**, 623 (1994).
[11] M. Moshe, H. Neuberger, and B. Shapiro, Phys. Rev. Lett. **73**, 1497 (1994).
[12] A. Bulatov, D. Schmeltzer, and J. Birman (unpublished).
[13] B. L. Altshuler, I. Kh. Zharekeshev, S. A. Kotochigova, and B. I. Shklovskii, Sov. Phys. JETP **67**, 625 (1988).
[14] U. Sivan and Y. Imry, Phys. Rev. B **35**, 6074 (1987).
[15] K. B. Efetov, Adv. Phys. **32**, 53 (1983).
[16] F. M. Izrailev, Phys. Rep. **196**, 299.
[17] R. W. Anderson, Phys. Rev. **109**, 1402 (1958).
[18] O. W. Greenberg, Phys. Rev. D **43**, 4111 (1991); M. Bozejko and R. Speicher, Commun. Math. Phys. **137**, 519 (1990).
[19] D. J. Gross and A. A. Migdal, Phys. Rev. Lett. **64**, 127 (1990); E. Brezin and V. Kazakov, Phys. Lett. B **236**, 144 (1990).
[20] E. Witten, Nucl. Phys. **B340**, 281 (1990).
[21] N. Rosenzweig and C. E. Porter, Phys. Rev. **120**, 1698 (1960).
[22] F. Leyvraz and T. H. Seligman, J. Phys. A **23**, 1555 (1990).
[23] B. I. Shklovskii, B. Shapiro, B. R. Sears, P. Lambrianides, and H. B. Shore, Phys. Rev. B **47**, 11 487 (1993).
[24] V. E. Kravtsov, I. V. Lerner, B. L. Altshuler, and A. G. Aronov, Phys. Rev. Lett. **72**, 88 (1994).
[25] C. Itzykson and J.-M. Drouffe, *Statistical Field Theory* (Cambridge University Press, Cambridge, 1992), p. 694.