Universality of Random-Matrix Results for Non-Gaussian Ensembles

G. Hackenbroich and H. A. Weidenmüller
Max-Planck-Institut für Kernphysik, 69117 Heidelberg, Germany
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We study random-matrix ensembles with a non-Gaussian probability distribution $P(H) \sim \exp[-N\mathrm{tr}\, V(H)]$, where N is the dimension of the matrix H and $V(H)$ is independent of N. Using Efetov's supersymmetry formalism, we show that in the limit $N \to \infty$ both energy level correlation functions and correlation functions of S-matrix elements are independent of $P(H)$ and hence universal on the scale of the local mean level spacing. This statement applies to each of the three generic ensembles (unitary, orthogonal, and symplectic). Universality is also found for correlation functions depending on some external parameter. Our results generalize previous work by Brezin and Zee [Nucl. Phys. **B402**, 613 (1993)].

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The energy levels of a variety of physical systems including complex nuclei, disordered conductors, and classically chaotic systems exhibit universal behavior: The statistical properties of the spectrum depend only on the fundamental symmetries of the underlying Hamiltonian and can be described by random-matrix theory [1]. Three symmetry classes exist: Systems with broken time-reversal symmetry are described by the unitary ensemble and time-reversal invariant systems by either the symplectic or the orthogonal ensemble depending on whether spin-orbit coupling is present or not. The symmetry of the Hamiltonian does not specify the random-matrix ensemble completely, however. In general, one requires, in addition, that the matrix elements be statistically independent from each other. This additional condition defines the Gaussian matrix ensembles which are used practically exclusively in applications of random–matrix theory.

The use of Gaussian ensembles, although mathematically convenient, is unsatisfactory from a physical point of view. This is true for several following reasons: (i) The assumption of statistical independence is not motivated by first principles. (ii) The Gaussian ensembles fail to describe global properties of the experimentally observed spectra. Indeed, they predict a universal form (the semicircle law for matrices of high dimension) for the mean level density $\rho(E)$, whereas this quantity is known to be system specific and nonuniversal. (iii) Imposing a given form of $\rho(E)$ and using a maximum-entropy approach, Balian [2] derived a non-Gaussian form for the generic random-matrix ensemble.

Numerical studies [3] have led to the conjecture [1] that the *local* spectral fluctuation properties of such non-Gaussian ensembles are independent of the measure, are identical to those of the Gaussian ensemble in the same symmetry class, and are, hence, universal. In the first analytical investigation of the influence of a non-Gaussian measure on the spectral statistic, Brezin and Zee [4] proved universality for the local two-point level correlation function for the unitary ensemble. (Properly smoothed level correlations on scales large compared to d were found to depend on the measure only through the end points of the spectrum. This latter result was later generalized to all three ensembles by Beenakker [5].)

It is the aim of this paper to prove (in the limit $N \to \infty$, where N denotes the dimension of the matrices) the universality of arbitrary local correlation functions for any non-Gaussian ensemble in any of the three symmetry classes. More specifically, we consider a correlation function C involving an arbitrary number of level-density factors and/or S-matrix elements. This function may depend parametrically on energy arguments and/or on additional parameters like the strength of an external magnetic field. We do allow for symmetry breaking. (An external magnetic field, for instance, breaks orthogonal symmetry.) We compare the correlation function C_G evaluated for the Gaussian ensemble, and its analog C_P evaluated for a non-Gaussian ensemble having the same symmetry and a distribution $P(H)$ defined as in Eq. (1) below. Under the single assumption that all parameters in C range over an interval Δ containing on average a finite number of levels $\Delta \sim \mathcal{O}(N^{-1})$, we show that $C_G(d_G(E)) = C_P(d_P(E))$. Here, $d_G(E)$ and $d_P(E)$ are the average local mean level spacings for the Gaussian and the non-Gaussian ensemble, respectively, both evaluated at energy E, which is located at the center of the interval Δ. These very general results are obtained by using the supersymmetry method. To allow for non-Gaussian probability measures, our derivation differs from the usual formulation of this method [6,7].

For definiteness we consider the unitary ensemble in the following. We emphasize that the orthogonal and symplectic cases can be treated along exactly parallel lines. Accordingly, we study an ensemble of $N \times N$ Hermitian random matrices H with volume element $d[H] = \prod_{i\geq j}^{N} d\mathrm{Re}H_{ij} \prod_{i>j}^{N} d\mathrm{Im}H_{ij}$. The probability density $P(H)$ is defined by

$$P(H) = Z^{-1} \exp\{-N \,\mathrm{tr}\, V(H)\}, \qquad (1)$$

where Z is a normalization constant. This is the most general density compatible with the basic assumption of

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random-matrix theory, namely, that $P(H)dH$ is invariant under unitary transformations $H \to UHU^{-1}$. The function V is assumed both to confine the spectrum to some finite interval and to generate a smooth mean level density, in the limit $N \to \infty$ [8]. Then, for the ensemble defined by Eq. (1), the mean level spacing d is of order N^{-1}. Note that $V(H) = gH^2$, $g > 0$, defines the Gaussian unitary ensemble.

In the supersymmetry method, we generically express [7] correlation functions as derivatives of a generating functional I. The detailed form of I depends on the particular correlation function under study. All generating functionals possess a common structure, however. Our proof relies exclusively on this common structure which we display by writing I in terms of an integral over a supervector Ψ with bosonic (commuting) and fermionic (anticommuting) components

$$I = \int d[\Psi] \left\langle \exp\left\{ \frac{i}{2} \Psi^\dagger \mathbf{L}^{1/2} (\mathbf{H} - \mathbf{E} + \mathbf{M}) \mathbf{L}^{1/2} \Psi \right\} \right\rangle, \tag{2}$$

where the brackets denote the ensemble average $\langle \cdots \rangle \equiv \int d[H] P(H) (\cdots)$. We exemplify the common structure and the content of Eq. (2) for the case of the two-point function, taken at energies E and E'. Here, one defines the supervector by $\Psi^T = (S_1^T, \chi_1^T, S_2^T, \chi_2^T)$ with complex bosonic entries S_1, S_2 and complex fermionic entries χ_1, χ_2, each entry being itself an N-dimensional vector. The measure has the form $d[\Psi] = \prod_{\mu=1}^{N} \prod_{j=1}^{2} i dS_{\mu j}^* \, dS_{\mu j} \, d\chi_{\mu j}^* \, d\chi_{\mu j}$. The Hamiltonian $\mathbf{H}$ is the direct product of the $N \times N$ Hamiltonian H and the unit matrix in the superspace. The energy $\mathbf{E}$ stands for the product of the mean energy $E = (E_1 + E_2)/2$ and the unit matrix in both the level space and superspace $\mathbf{L}$ is the direct product of the unit matrix in level space with $L = \mathrm{diag}(1, 1, -1, -1)$. The matrix $\mathbf{M}$ contains energy differences and the source terms. In the case of scattering problems, $\mathbf{M}$ will also contain couplings to external channels. To account for dependences on external parameters, $\mathbf{M}$ may contain additional random matrices besides H over which additional ensemble averages must be performed. (We postpone this calculation and confine attention first to the ensemble average over H.) It is a central property of Eq. (2) that $\mathbf{M} = O(N^{-1})$. This property comes about because we are interested in correlations involving energies *of the order of the mean level spacing* $d \sim N^{-1}$. For n-point functions with $n > 2$, the *form* of Eq. (2) remains unchanged, although the dimensions of the vectors Ψ, $\Psi^\dagger$ and of the matrices $\mathbf{H}$, $\mathbf{E}$, $\mathbf{L}$, $\mathbf{M}$ in superspace will increase. Our proof applies to all these cases because it is independent of these dimensions.

In the Gaussian case, one usually decouples the interaction generated by the ensemble average by means of a Hubbard-Stratonovich transformation [7]. This procedure introduces a supermatrix σ and maps the generating functional onto a nonlinear σ model. The procedure relies on the Gaussian form of the probability density and does not apply to general $P(H)$. The central point of our argument is based on the observation that it is nevertheless possible to introduce the "composite variables" σ for any $P(H)$. Indeed, for any $P(H)$ the unitary invariance of the ensemble implies that for $N \to \infty$, the integrand in Eq. (2) depends on Ψ and $\Psi^\dagger$ only via the invariant form $A_{\alpha\beta} = N^{-1} L_{\alpha\gamma}^{1/2} \sum_{\mu=1}^{N} \Psi_{\mu\gamma} \Psi_{\mu\delta}^\dagger L_{\gamma\beta}^{1/2}$. Here μ, ν are level indices and α, β, γ, δ superindices [9]. We explicitly introduce a supermatrix σ with the same dimension and symmetry properties as A by writing I as an integral over a δ function,

$$I = \int d[\Psi] \int d\sigma \; \delta(\sigma - A) \times \left\langle \exp\left\{ \frac{i}{2} \Psi^\dagger \mathbf{L}^{1/2} \mathbf{G} \mathbf{L}^{1/2} \Psi \right\} \right\rangle, \tag{3}$$

with the abbreviation $\mathbf{G} \equiv \mathbf{H} - \mathbf{E} + \mathbf{M}$. The δ function is replaced by its Fourier representation

$$I = \int d[\Psi] \int d\sigma \int d\tau \, \exp\left\{ \frac{i}{2} N \, \mathrm{trg}(\tau\sigma) \right\} \times \left\langle \exp\left\{ \frac{i}{2} \Psi^\dagger \mathbf{L}^{1/2} (\mathbf{G} - \tau) \mathbf{L}^{1/2} \Psi \right\} \right\rangle, \tag{4}$$

and the multiple Gaussian integral over the Ψ supervector is performed

$$I = \int d\sigma \int d\tau \exp\left\{ \frac{i}{2} N \, \mathrm{trg}(\tau\sigma) \right\} \times \left\langle \exp\left[-\frac{1}{2} \, \mathrm{tr\,trg} \, \ln[\mathbf{G} - \tau] \right] \right\rangle, \tag{5}$$

where trg denotes the supertrace (the definition of the supertrace can be found in Ref. [7]). We have now expressed the functional I as a integral over two coupled supermatrices σ and τ which contain all relevant degrees of freedom. In the limit $N \to \infty$, the remaining integrals can be done explicitly using the saddle-point approximation. In particular, it will turn out that the (diagonal) saddle point of the σ integral determines the mean level density. These steps will prove the claimed universality by comparison with the well-known Gaussian case.

To perform the ensemble average we transform H to diagonal form, $(UHU^\dagger) = \Lambda$, and integrate separately over eigenvalues Λ and eigenvectors U. Expanding in powers of $\mathbf{M}$, we have

$$\left\langle \exp\left[-\frac{1}{2} \, \mathrm{tr\,trg} \, \ln[\mathbf{G} - \tau] \right] \right\rangle = \left\langle \exp\left[-\frac{1}{2} \, \mathrm{tr\,trg} \, \ln D \right] \exp\left[-\frac{1}{2} \, \mathrm{tr\,trg} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (D^{-1} U^\dagger \mathbf{M} U)^n \right] \right] \right\rangle, \tag{6}$$

where $D \equiv \Lambda - E - \tau$ is diagonal in the level space. The expansion in powers of $\mathbf{M}$ in Eq. (6) cannot be terminated with the first-order term, because any power of $\mathbf{M}$ may be of the same order N^{-1} as $\mathbf{M}$. This is the case for S-matrix correlation functions [7]. The distribution of eigenvectors does not depend on the form of the probability density in Eq. (1). In the large-N limit the eigenvectors are Gaussian distributed [3] and the average over eigenvectors is evaluated using Wick contractions. To leading order in powers of N^{-1} we find that the last exponential in Eq. (6) takes the simple form $-\frac{1}{2}\,\mathrm{tr}\,\mathrm{trg}[1 + (N^{-1}\mathrm{tr}D^{-1})\mathbf{M}]$. The remaining eigenvalue integrations are done using the saddle-point approximation [10]. Explicitly, the average over eigenvalues appearing in Eq. (6) involves the exponential

$$-\frac{1}{2}\sum_{\mu}\mathrm{trg}\ln(D)_{\mu} - \frac{1}{2}\,\mathrm{tr}\,\mathrm{trg}\,\ln\left[1 + \left(\frac{1}{N}\sum_{\mu}(D^{-1})_{\mu}\right)\mathbf{M}\right] - N\sum_{\mu}V(\lambda_{\mu}) + 2\sum_{\mu<\nu}\ln|\lambda_{\mu} - \lambda_{\nu}|\,, \quad (7)$$

where the last term results from the Jacobian associated with the transformation from the matrix elements of H to its eigenvalues. Of the four terms in expression (7) the first is $\mathcal{O}(N)$ and the second $\mathcal{O}(1)$. The last two terms are $\mathcal{O}(N^2)$ and determine the saddle-point values $\lambda_{\mu}^{\mathrm{sp}}$. The calculation is explicitly carried out in Ref. [10] and introduces the average local level density $\rho(E)$, and the resolvent F, defined by

$$\frac{1}{N}\sum_{\mu}\frac{1}{E + \tau - \lambda_{\mu}^{\mathrm{sp}}} \xrightarrow{N\to\infty} \int dE'\frac{\rho(E')}{E + \tau - E'} \equiv F(E + \tau). \quad (8)$$

Substituting $\lambda_{\mu}^{\mathrm{sp}}$ for λ_{μ} in the first two terms of Eq. (7), expanding the terms $\sim\mathcal{O}(N^2)$ around the saddle point, and performing the Gaussian integrals, we find

$$I = \int d\sigma \int d\tau \times \exp\left\{\frac{i}{2}N\mathrm{trg}(\tau\sigma) - \frac{1}{2}\sum_{\mu}\mathrm{trg}\ln[\lambda_{\mu}^{\mathrm{sp}} - E - \tau]\right\} \times \exp\left\{-\frac{1}{2}\,\mathrm{tr}\,\mathrm{trg}\,\ln[1 + F(E + \tau)\mathbf{M}]\right\}. \quad (9)$$

(The integration over the λ_{μ} around the saddle point cancels against the normalization Z.) Again, the saddle-point approximation is used to integrate over τ. The last exponential in Eq. (9) has a term of order $\mathcal{O}(1)$ in the exponent and can be omitted. Hence, for fixed σ the equation $i\sigma = F(E + \tau^{\mathrm{sp}})$ determines the saddle point $\tau^{\mathrm{sp}}(\sigma, E)$. By expanding the exponent to quadratic order in the fluctuations $\delta\tau$, one can easily verify that the integral over $\delta\tau$ yields unity. Therefore, we obtain

$$I = \int d\sigma \times \exp\left\{\frac{i}{2}N\mathrm{trg}(\sigma\tau^{\mathrm{sp}}) - \frac{1}{2}\sum_{\mu}\mathrm{trg}\ln(\lambda_{\mu}^{\mathrm{sp}} - E - \tau^{\mathrm{sp}})\right\} \times \exp\left\{-\frac{1}{2}\,\mathrm{tr}\,\mathrm{trg}\ln(1 - i\sigma\mathbf{M})\right\}, \quad (10)$$

where now only the integration over the supermatix σ remains to be done. The saddle point σ^{sp} is found from the first two terms in the exponent

$$i\sigma^{\mathrm{sp}}\frac{\partial\tau^{\mathrm{sp}}}{\partial\sigma}\bigg|_{\sigma^{\mathrm{sp}}} + i\tau^{\mathrm{sp}} = F(E + \tau^{\mathrm{sp}})\frac{\partial\tau^{\mathrm{sp}}}{\partial\sigma}\bigg|_{\sigma^{\mathrm{sp}}}. \quad (11)$$

The saddle-point equations for τ and σ together show that $\tau^{\mathrm{sp}}(\sigma^{\mathrm{sp}}) = 0$ and $i\sigma^{\mathrm{sp}} = F(E)$. One observes that the saddle-point equation for σ^{sp} is invariant under pseudounitary transformations. This implies that also $\sigma_{\mathrm{G}} = T^{-1}\sigma^{\mathrm{sp}}T$ is a saddle point, where T generates pseudounitary transformation on the space of supermatrices. In general, $\sigma_{\mathrm{G}} \neq \sigma^{\mathrm{sp}}$ and hence the σ_{G} form a manifold of solutions of the saddle-point equation. We expand the exponential in Eq. (10) in the vicinity of the saddle-point solution σ_{G}, carry out the integral over the massive modes (which gives unity), and find the result

$$I = \int d\mu(t)\exp\left\{-\frac{1}{2}\,\mathrm{tr}\,\mathrm{trg}\,\ln[1 - T^{-1}F(E)T\mathbf{M}]\right\}, \quad (12)$$

where the integration is now over the manifold of saddle points. As usual [7] one has to give E an imaginary part such that $\mathrm{Im}F(E) \sim L$ to guarantee convergence. Both the structure of the saddle-point manifold and the measure $d\mu$ depend only on the symmetry of the ensemble and on the dimension of the supervectors Ψ, $\Psi^{\dagger}$ in Eq. (2). In particular, both are independent of the probability density $P(H)$. On the other hand, the latter specifies the mean level density $\rho(E)$ and the function $F(E)$ and hence sets the local energy scale. To see this most clearly one chooses E such that $\mathrm{Re}F(E) = 0$ and therefore $F(E) = -i\pi\rho(E)L$ [11]. For a symmetric confining potential V one may take E at the center of the spectrum $E = 0$. Then

$$I = \int d\mu(t)\exp\left\{-\frac{1}{2}\,\mathrm{tr}\,\mathrm{trg}\,\ln[1 + i\pi\rho(E)T^{-1}LT\mathbf{M}]\right\}. \quad (13)$$

This proves universality: Our result has the same form as in the Gaussian case [7], the probability density enters only through the local mean level density $\rho(E)$. On the scale set by $\rho(E)$ all derivatives of I and hence all correlation functions are independent of the probability density $P(H)$ and are thus universal.

The proof, presented here for the unitary ensemble, applies equally to the orthogonal and the symplectic ensemble. In either of the latter cases, the structure of the supervector Ψ differs from the unitary case. This structure is reflected in the symmetry properties and

dimensions of the matrices σ and τ and, eventually, of the matrix T which generates transformations on the saddle-point manifold. However, our threefold use of the saddle-point approximation is completely independent of such symmetry properties.

Last we turn to correlation functions which depend on an external parameter [12]. One has to distinguish between two cases The external perturbation may either preserve [case (i)] or violate [case (ii)] the symmetry of the original matrix ensemble. We consider case (i) first and demonstrate our point for the case of the two-point level correlation function. It is defined by $\langle \mathrm{tr}(E + \epsilon/2 - H - \sqrt{\alpha/N}H')\mathrm{tr}(E - \epsilon/2 - H + \sqrt{\alpha/N}H')\rangle$, where the energy difference ϵ is $\mathcal{O}(N^{-1})$ and H' is a matrix ensemble with the same symmetry and the same distribution function as the original ensemble H. The average is over the distributions of both H and H'. For simplicity we restrict ourselves to symmetric distributions. The factor $\sqrt{N^{-1}}$ appearing in the definition of the two-point correlation function ensures that the correlations decay on the typical scale $\alpha \sim \mathcal{O}(1)$. In the generating functional of Eq. (2) there now appears an additional term $\sqrt{\alpha/N}H'L$ which can be included in the definition of $\mathbf{M}$ by replacing $\mathbf{M} \to \mathbf{M} + \sqrt{\alpha/N}H'L$. After averaging over the ensemble H one finds that I is given by

$$I = \int d\mu(t) \times \left\langle \exp\left\{-\frac{1}{2}\,\mathrm{tr}\,\mathrm{trg}\,\ln[1 + i\pi\rho(E)T^{-1}LT\mathbf{M}]\right\}\right\rangle_{H'}. \tag{14}$$

Now the logarithm is expanded in powers of H'. Because of the factor $\sqrt{\alpha/N}$, taking the ensemble average over H' reduces to calculating the second cumulant, all higher-order cumulants being small in comparison by at least a factor $N^{-1/2}$. The final result for I is

$$I = \int d\mu(t) \exp\left\{-\frac{1}{2}\,\mathrm{tr}\,\mathrm{trg}\,\ln[1 + i\pi\rho(E)T^{-1}LT\mathbf{M}]\right\} \times \exp\left\{-\frac{\alpha}{N}\left(\frac{\pi\rho(E)}{2}\right)^2[\mathrm{trg}(T^{-1}LTL)^2\langle \mathrm{tr}\,(H')^2\rangle_{H'}]\right\}. \tag{15}$$

The distribution over H' enters only through its second moment. Its value for a non-Gaussian distribution for H' differs from what one would find for a Gaussian distribution. However, this difference does not affect the form of the correlation function and only leads to a rescaling of the parameter α. Case (ii) is treated along exactly parallel lines and leads to exactly the same conclusion: Aside from a scaling factor affecting the parameter which governs symmetry breaking, the form of the correlation function is the same for Gaussian and non-Gaussian ensembles.

In summary, we have investigated the consequences of non-Gaussian probability measures within random-matrix theory. We have shown that in the limit of a large number of levels global and local properties of the spectrum separate. Global properties like the mean level density do depend on the form of the measure. Local properties, in contrast, are independent of the measure. They are determined only by the symmetry of the ensemble, and they are identical to those of the corresponding Gaussian ensemble. This holds for all three generic ensembles and for arbitrary form of the measure. Our analytical result establishes generally and for the first time that all local random-matrix correlations are independent of the measure and hence universal.

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