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# Quantum-State Tomography and Discrete Wigner Function

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A tomographical scheme is proposed to infer the quantum states of finite–dimensional systems from experiments. For this a new discrete Wigner formalism is developed.

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As a fundamental feature of quantum mechanics we cannot see physical objects *as they are* since the overall backaction of any observation cannot be made much less than Planck's constant $\hbar$. Instead, we see only the various aspects of the physical objects, like the wave or the particle aspects which depend on the particular kind of observation. In this respect we are really like the prisoners in Plato's famous parable [1] who were chained in a cave and forced to see only the shadows of the things outside but not the things as they are. Can we infer the things from their shadows? Tomography is a method for building up a picture of a hidden object from various observable projections. For instance, computer assisted tomography gives insight into a living body by evaluating recorded transmission profiles of radiation which has penetrated the body from various directions. In quantum optics tomography was recently applied to reconstruct the Wigner function of a light beam from measured data [2]. The Wigner function [3]

$$W(x,p) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \exp(2ipy)\langle x - y|\hat{\rho}|x + y\rangle\, dy \quad (1)$$

is a quasiprobability distribution for the position $x$ and the momentum $p$ of a mechanical system. (The kets $|x\rangle$ denote the position eigenstates and $\hat{\rho}$ is the density matrix.) It characterizes the quantum object *as it is* since it determines the quantum state $\hat{\rho}$ uniquely. The observed quantities $x_\theta$ (quadratures) are mixtures of position and momentum $x_\theta = x\cos\theta + p\sin\theta$ which reflect the various quantum aspects of a mechanical system. They may occur as the amplitudes of a harmonic oscillator with the phases $\theta$. Quadrature histograms $w_\theta(x_\theta)$ are projections (Radon transformations) of the Wigner function (1)

$$w_\theta(x_\theta) = \int_{-\infty}^{+\infty} W(x_\theta\cos\theta - p_\theta\sin\theta, x_\theta\sin\theta + p_\theta\cos\theta)\, dp_\theta\,. \quad (2)$$

From the set of histograms $w_\theta(x_\theta)$ the Wigner function itself can be reconstructed via the inverse Radon transformation, as was shown in a fundamental paper by Vogel and Risken [4]. Apart from the originally intended quantum–optical applications [2] tomography may also serve as an experimental tool for reconstructing scalar de Broglie fields [5] or molecular wave packets [6].

Can we use tomography to measure the internal states of quantum fields? Can we measure the quantum states of finite discrete systems like atoms or spins [78–12]? For this we could transcribe the continuous Wigner formalism for discrete quantum mechanics. Some time ago Wootters [13] pioneered a Wigner formalism for finite systems with, however, prime dimension. Cohendet *et al.* [14] considered Wigner functions for odd–dimensional systems. In this Letter the problem is solved in full generality for arbitrary dimensions. Discrete Wigner functions are designed in such a way that their projections can be observed experimentally. The quantum state is inferred from the measured histograms. The formalism is general and abstract, but to have a physical picture in mind let us imagine as our system a spin or angular momentum with fixed square total spin or angular momentum, respectively. The phase space is spanned by an angular momentum or spin component $m$ and by the phase $\mu$ instead of the position $x$ and the momentum $p$. Since in the finite–dimensional case phases are discrete, the phase space contains discrete points. (It will be defined as a discrete lattice.) The transcription of the continuous Wigner formalism for discrete quantum mechanics involves some interesting subtleties like the transcription of a symphony for a chamber orchestra. Most remarkably, odd– and even–dimensional systems (angular momenta and spins, *bosons* and *fermions*) must be distinguished [15]. This might be related to the fact that spins require a $4\pi$ rotation to return to the initial state while $2\pi$ is sufficient for angular momenta.

*Definition.*—Suppose a quantum object with density matrix $\hat{\rho}$ in a $d$–dimensional space. The basis vectors $|m\rangle$ are labeled by integers $m$ with the convention $|m + d\rangle = |m\rangle$ like in the Pegg–Barnett formalism [16]. We define the *characteristic function*

$$\tilde{W}(n,\nu) \equiv \sum_{k=0}^{d-1} \exp\left[-\frac{4\pi}{d}\, n(k + \nu)\right] \langle k|\hat{\rho}|k + 2\nu\rangle \quad (3)$$

and the *discrete Wigner function*

$$W(m,\mu) \equiv \frac{1}{D^2} \sum_{n,\nu} \exp\left[\frac{4\pi i}{d}\,(nm + \nu\mu)\right] \tilde{W}(n,\nu). \quad (4)$$

In the odd–dimensional case (for bosons) the phase space consists of integers $(m,\mu)$, $D$ equals $d$, and whenever not explicitly stated all summations are to be carried out from $-\frac{1}{2}(d-1)$ to $\frac{1}{2}(d-1)$. For mathematical subtleties fermions require also half odds to be placed between the integers in the phase space $(m,\mu)$ [17]. Let us understand all fermionic variables as integers and half odds ranging from $-\frac{1}{2}d$ to $\frac{1}{2}(d-1)$ whenever not explicitly stated, and $D$ equals $2d$. In any case, the discrete Wigner function (4) is real, normalized to unity, and periodic $W(m + d,\mu) = W(m,\mu + d) = W(m,\mu)$. Substituting the definition (3) of the characteristic function into Eq. (4) we find a familiar–looking formula for the Wigner function

$$W(m,\mu) = \frac{1}{D} \sum_{n} \exp\left(\frac{4\pi i}{d}\, n\mu\right) \langle m - n|\hat{\rho}|m + n\rangle, \quad (5)$$

with the convention for fermions that states $|k\rangle$ labeled by half–odd $k$ are regarded as being zero [18]. From this expression the *overlap relation*

$$\mathrm{Tr}\{\hat{F}_1\hat{F}_2\} = D \sum_{m,\mu} W_1(m,\mu)\, W_2(m,\mu) \quad (6)$$

is easily derived for arbitrary operators $\hat{F}_1$ and $\hat{F}_2$ and their corresponding Wigner functions $W_1$ and $W_2$. The overlap relation (6) provides us with the key for calculating expectation values via Wigner functions or, more generally, for formulating quantum mechanics without probability amplitudes [19].

*Quadratures.*—We define quadratures by the discrete Radon transformation

$$w(m;a,b) \equiv \sum_{\mu} W(am - b\mu, bm + a\mu), \quad (7)$$

with the integers $a$ and $b$. The formula (7) looks like approximating $\cos\theta$ and $\sin\theta$ by rational numbers in the continuous Radon transformation (2). In view of Plato's parable [1] we may call it a *Plato transformation*, provided of course that the $w(m;a,b)$ are really observable quantities—quantum shadows in Plato's sense. Now we prove that this is indeed the case which requires a bit of elementary number theory to consider. Suppose that $a$ and $b$ have no common divisor and that $b \neq 0$. We substitute the definition (4) of the Wigner function in Eq. (7), sum over $\mu$, and obtain

$$w(m;a,b) = \frac{1}{D} \sum_{n,\nu} \exp\left[\frac{4\pi i}{d}\, m(an + b\nu)\right] \tilde{W}(n,\nu), \quad (8)$$

with

$$nb = \nu a (\mathrm{mod}\, d). \quad (9)$$

We study the condition (9) in more detail. Let $z$ be the greatest common divisor of $d$ and $b$ ($z = 1$ if there is not any). We define

$$d = d'z, \qquad D = D'z, \qquad b = b'z. \quad (10)$$

The condition (9) implies that

$$\nu = \nu' z. \quad (11)$$

Representing

$$n = n' + n'' d', \quad (12)$$

we obtain from Eq. (9)

$$n'b' = \nu' a (\mathrm{mod}\, d'), \quad (13)$$

while $n''$ is left unfixed. Inserting the definition (3) of the characteristic function into Eq. (8) and summing over the $n''$ values gives

$$w(m;a,b) = \frac{1}{D'} \sum_{\nu'} \sum_{k=0}^{d-1} \times \exp\left[\frac{4\pi i}{d}\,[n'(am - k - \nu' z) + \nu' z b m]\right] \times \langle k|\hat{\rho}|k + 2\nu' z\rangle \quad (14)$$

together with the condition

$$2(am - k - \nu' z) = 0 (\mathrm{mod}\, z). \quad (15)$$

Now we rearrange the summations in Eq. (14), utilizing the periodicity of the matrix elements and of the phase factors. Bosons and fermions must be treated separately.

*Bosons.*—There are always two integers $k_1'$ and $k_2'$ with $k - am = 2b'k_1' \pmod d$ and $k - am + 2\nu = 2b'k_2' \pmod d$ [20]. We get $am - k - \nu = -b'(k_1' + k_2') \pmod d$ together with $\nu = b'(k_2' - k_1') \pmod d$, and the conditions (11) and (13) imply that $k_1' = k_1 z$ and $k_2' = k_2 z$. Substituting this into Eq. (14) and observing the condition (13) we obtain finally

$$w(m;a,b) = \langle m;a,b|\hat{\rho}|m;a,b\rangle, \tag{16}$$

with

$$|m;a,b\rangle = (zd)^{-1/2}\sum_k \exp\left[\frac{4\pi i}{d} bk(bm - ak)\right] \times |am + 2bk\rangle, \tag{17}$$

which proves that the discrete quadratures are indeed observable quantities. The components of the quadrature eigenstates $\langle n|m;a,b\rangle$ appear as appropriate discrete versions of the harmonic–oscillator Green's function [21], as we would expect from the correspondence to the continuous Wigner formalism. The marginals of the Wigner function $W(m,\mu)$ with respect to $m$ are the discrete phase distributions with the Pegg–Barnett phase states [16]

$$|\mu;0,1\rangle = d^{-1/2}\sum_k \exp\left(\frac{2\pi i}{d}\mu k\right)|k\rangle. \tag{18}$$

This supports the interpretation of $\mu$ as a phase with respect to the angular momentum $m$. An elementary calculation yields

$$|m;1,0\rangle = |m\rangle, \tag{19}$$

i.e., the marginals of $W(m,\mu)$ with respect to $\mu$ are the angular–momentum histograms.

*Fermions.*—The introduction of the half odds as possible values for the phase–space variables $(m,\mu)$ means that the Wigner function contains redundant information. We would expect that quadrature histograms are redundant as well, since otherwise more measurements than necessary were required to reconstruct the quantum state. Suppose that $a$ and $b$ are both odd. (Since $a$ and $b$ have no common factor, we have excluded the case that both might be even.) The condition (9) implies that then $an + b\nu$ must be an integer, and Eq. (8) leads to

$$w(m;a,b) = w\left(m + \tfrac{1}{2}d;a,b\right) \quad \text{for } a,b \text{ odd}. \tag{20}$$

When $a$ is even and $b$ is odd Eq. (9) implies that $n$ must be an integer. The definition (3) of the characteristic function gives $\exp\{4\pi i m[an + b(\nu + \frac{1}{2}d)]/d\}\tilde{W}(n,\nu + \frac{1}{2}d) = (-1)^{2m}\exp[4\pi i m(an + b\nu)/d]\tilde{W}(n,\nu)$. If $m$ is half odd this term cancels with $\exp[4\pi i m(an + b\nu)/d]\tilde{W}(n,\nu)$ in the summation in Eq. (8). In this case the quadratures vanish. We obtain the same result if $a$ is odd and $b$ is even, so that

$$w(m;a,b) = 0 \quad \text{for } a - b \text{ odd}, \quad m \text{ half odd}. \tag{21}$$

The relations (20) and (21) show clearly that the quadrature histograms have maximal $d$ independent components. Now we proceed in a similar way as in the bosonic case. There are always two integers or half odds $k_1'$ and $k_2'$ to express $k - am = b'k_1' \pmod d$ and $k - am + 2\nu = b'k_2'$ when $am$ is an integer or half odd, respectively. We obtain again that quadratures are observable quantities and find the following compact expression for the quadrature eigenstates:

$$|m;a,b\rangle = (zd)^{-1/2}\sum_k \exp\left[\frac{\pi i}{d} bk(2bm - ak)\right] \times |am + bk\rangle, \tag{22}$$

with the convention that the summation is to be performed over all integers and half odds between $-\frac{1}{2}d$ and $\frac{1}{2}(d-1)$ and that states labeled by half odds are regarded as being zero. Additionally, we must require that

$$|m;a,b\rangle = 0 \quad \text{for } a - b \text{ odd}, \quad m \text{ half odd}. \tag{23}$$

Again, the fermionic quadrature eigenstates are discrete versions of the harmonic–oscillator Green's function [21]. They almost coincide with the bosonic states (17). Marginals of the Wigner function yield spin or phase histograms, respectively, with the important condition, however, that they vanish for half–odd arguments [18]. In this way the introduced half odds in the phase space $(m,\mu)$ do not appear as directly observable quantities. They are "ghost variables" which are nevertheless necessary for mathematical consistency [17].

*Measurement.*—Discrete quadratures are observable in principle. How can they be measured in practice? For this the system should be unitarily transformed in such a way that the quadrature eigenstates $|m;a,b\rangle$ become the energy levels. Any kind of population measurement yields then the quadrature histogram $w(m;a,b)$. The transformation can be done by Ramsey techniques. Ramsey fields create arbitrary quantum superpositions of two levels in a controlled way. Any superposition of a multitude of levels can be produced by a sequence of Ramsey zones (in a similar way as any optical multiport can be realized by a sequence of beam splitters [22]).

*Inversion.*—Given the quadrature histograms, how can we reconstruct the quantum state? Let us Fourier transform the quadrature distribution

$$\tilde{w}(n;a,b) = \sum_m \exp\left(-\frac{4\pi i}{d}mn\right) w(m;a,b). \quad (24)$$

Substituting instead of $w(m;a,b)$ the defining Plato transformation (7) and comparing the result with the characteristic function expressed in terms of the Wigner function

$$\tilde{W}(n,\nu) = \sum_{m,\mu} \exp\left[-\frac{4\pi i}{d}(mn + \mu\nu)\right] W(m,\mu) \quad (25)$$

yields

$$\tilde{w}[n(a^2 + b^2);a,b] = \tilde{W}(na,nb). \quad (26)$$

The Fourier–transformed quadrature histograms appear as the characteristic function in "discrete polar coordinates" quite similar indeed to the continuous case [4]. Given the characteristic function, the Wigner function is determined by its very definition (4). The density matrix is easily obtained from the characteristic function (3) by discrete Fourier transformation

$$\langle m - b|\hat{\rho}|m + b\rangle = \frac{1}{d}\sum_{a=0}^{d-1} \exp\left(\frac{4\pi i}{d}am\right)\tilde{W}(a,b) \quad (27)$$

for bosons and

$$\left\langle m - \tfrac{1}{2}b|\rho\hat{M}|m + \tfrac{1}{2}b\right\rangle = \frac{1}{d}\sum_{a=0}^{d-1} \exp\left(\frac{2\pi i}{d}am\right)\tilde{W}\left(\tfrac{1}{2}a, \tfrac{1}{2}b\right) \quad (28)$$

for fermions. We may call the whole procedure the *inverse Plato transformation* because it is a way to infer the hidden quantum state from observations which represents the various aspects of the quantum object—the observable shadows.

The proposed method for quantum–state measurement offers the great practical possibility of gaining the maximal information about an unknown physical object allowed by the very principles of quantum mechanics. It can serve as an important experimental tool wherever quantum effects of discrete systems are of interest, like in atomic physics, atomic optics, nuclear physics, and in the physics of artificial atoms in semiconductor heterostructures. The basic theoretical tool—the discrete Wigner formalism—may share the fate of the continuous Wigner function which "was found by L. Szilard and [E. P. Wigner] some years ago for another purpose" [3]. It may find wide applications in the foundations of quantum mechanics, semiclassical quantum mechanics, thermodynamics, theoretical quantum optics, atomic, and nuclear physics.

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