Conditional Quantum Dynamics and Logic Gates

Adriano Barenco, David Deutsch, and Artur Ekert
Clarendon Laboratory, Physics Department, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom

Richard Jozsa
School of Mathematics and Statistics, University of Plymouth, Plymouth PL4 8AA, United Kingdom
(Received 21 September 1994)

Quantum logic gates provide fundamental examples of conditional quantum dynamics. They could form the building blocks of general quantum information processing systems which have recently been shown to have many interesting nonclassical properties. We describe a simple quantum logic gate, the quantum controlled-NOT, and analyze some of its applications. We discuss two possible physical realizations of the gate, one based on Ramsey atomic interferometry and the other on the selective driving of optical resonances of two subsystems undergoing a dipole-dipole interaction.

PACS numbers: 89.70.+c, 03.65.Bz, 03.75.Dg

The fact that quantum-mechanical processes, in principle, allow new types of information processing has been known for almost a decade [1,2]. Bennett and Wiesner have shown that the capacity of quantum channels can be doubled [3] and recent progress in quantum complexity theory [4] indicates that the computational power of quantum computers exceeds that of Turing machines. Hence the experimental realization of such processes is a most interesting issue. In this paper we concentrate on the basic constituents of any quantum information processing device, namely, quantum logic gates. We wish to stress the appearance of a *conditional quantum dynamics*, in which one subsystem undergoes a coherent evolution that depends on the quantum state of another subsystem. The unitary evolution operator for the combined system has the form

$$U = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1 + \cdots + |k\rangle\langle k| \otimes U_k\,, \tag{1}$$

where the projectors refer to quantum states of the *control* subsystem and the unitary operations U_i are performed on the *target* subsystem. The simplest nontrivial operation of this sort is the *quantum controlled-NOT*. We describe this gate, analyze some of its applications, and discuss physical realizations.

The *classical* controlled-NOT gate is a reversible logic gate operating on two bits ϵ_1 and ϵ_2; ϵ_1 is called the control bit and ϵ_2 the target bit. The value of ϵ_2 is negated if $\epsilon_1 = 1$, otherwise ϵ_2 is left unchanged. In both cases the control bit ϵ_1 remains unchanged. We define the *quantum* controlled-NOT gate C_{12} as that which effects the unitary operation on two qubits (two-state quantum systems), which in a chosen orthonormal basis $\{|0\rangle, |1\rangle\}$ in $\mathcal{H}_2$ reproduces the controlled-NOT operation,

$$|\epsilon_1\rangle|\epsilon_2\rangle \xrightarrow{C_{12}} |\epsilon_1\rangle|\epsilon_1 \oplus \epsilon_2\rangle\,, \tag{2}$$

where $\oplus$ denotes addition modulo 2. Here and in the following the first subscript of C_{ij} always refers to the control bit and the second to the target bit. Thus, for example, C_{21} performs the unitary operation defined by

$$|\epsilon_1\rangle|\epsilon_2\rangle \xrightarrow{C_{21}} |\epsilon_1 \oplus \epsilon_2\rangle|\epsilon_2\rangle\,. \tag{3}$$

The quantum controlled-NOT must be distinguished from the classical controlled-NOT which is performable on existing computers. The quantum controlled-NOT is a coherent operation on quantum states of the two qubits. The unitary operation defined by (2) is not the only one which reproduces the classical controlled-NOT on the computation basis states $|0\rangle$ and $|1\rangle$. We may introduce extra phases, and the most general such quantum operation is

$$|\epsilon_1\rangle|\epsilon_2\rangle \longrightarrow \exp(i\theta_{\epsilon_1\epsilon_2})|\epsilon_1\rangle|\epsilon_1 \oplus \epsilon_2\rangle\,. \tag{4}$$

This phase would be irrelevant to classical operations but gives rise to a family of inequivalent quantum gates.

Equations (2) or (3) define the gate C_{12} with respect to a specific basis, the computation basis $\{|0\rangle, |1\rangle\}$. It is also useful to consider generalizations of C_{12} which have the analogous effect on the control and target bits in bases that are different from the computation basis and possibly from each other. For example, C_{12} with respect to the basis $\{\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\}$ (for both qubits) is easily shown to be identical to C_{21} with respect to the basis $\{|0\rangle, |1\rangle\}$, i.e., the roles of the qubits are reversed by this simple change of basis. In the following, we always use the computation basis unless otherwise stated.

The quantum controlled-NOT gate has a variety of interesting properties and applications.

(1) C_{12} transforms superpositions into entanglements

$$C_{12} : (a|0\rangle + b|1\rangle)|0\rangle \longleftrightarrow a|0\rangle|0\rangle + b|1\rangle|1\rangle\,. \tag{5}$$

Thus it acts as a *measurement gate* because if the target bit ϵ_2 is initially in state $|0\rangle$ then this bit together with the gate amount to an apparatus that performs a perfectly accurate nonperturbing (quantum nondemolition [5]) measurement of ϵ_1.

(2) This transformation of superpositions into entanglements can be reversed by applying the same controlled-NOT operation again. Hence it can be used to implement

the so-called *Bell measurement* [6] on the two bits by disentangling the Bell states. From the four Bell states we get four product states:

$$C_{12}\frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)|0\rangle, \quad (6)$$

$$C_{12}\frac{1}{\sqrt{2}}(|0\rangle|1\rangle \pm |1\rangle|0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)|1\rangle. \quad (7)$$

Thus the Bell measurement on the two qubits is affected by a sequence of two independent two-dimensional measurements: in the computation basis for the target qubit and in the basis $\{\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\}$ for the control qubit. The realization of the Bell measurement is the main obstacle to the practical implementation of quantum teleportation [7] and dense quantum coding [3].

(3) Quantum state swapping can be achieved by cascading three quantum controlled-NOT gates,

$$C_{12}C_{21}C_{12}|\psi\rangle|\phi\rangle = |\phi\rangle|\psi\rangle, \quad (8)$$

for arbitrary states $|\psi\rangle$ and $|\phi\rangle$ (see also [8]).

(4) The quantum controlled-NOT gate may also be used to swap distantly separated states in the presence of a channel carrying only classical information. This is in contrast to the state swapping described above which requires the gate to be applied to the two states as inputs, so that they cannot be distantly separated at the time. Suppose that Alice and Bob, distantly separated, have states $|\alpha\rangle$ in $\mathcal{H}_0$ and $|\beta\rangle$ in $\mathcal{H}_5$, respectively, which they wish to swap (the identities of the states are assumed to be unknown to Alice and Bob). To achieve this they will need, on a previous occasion when they were close together or had access to a quantum communication channel, to have shared two pairs of qubits, one in the state $\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$ in $\mathcal{H}_1 \otimes \mathcal{H}_3$ and the other in the same maximally entangled state in $\mathcal{H}_2 \otimes \mathcal{H}_4$. States in $\mathcal{H}_0, \mathcal{H}_1$, $\mathcal{H}_2$ are localized near Alice, and states in $\mathcal{H}_3$, $\mathcal{H}_4$, $\mathcal{H}_5$ are localized near Bob. Let $\mathcal{M}$ denote a complete measurement in the computational basis $\{|0\rangle, |1\rangle\}$.

To swap $|\alpha\rangle$ and $|\beta\rangle$ Alice and Bob carry out the following protocol. *Step 1*: Alice performs C_{10} and then C_{02} while Bob performs C_{54} and then C_{35}. *Step 2*: Alice measures $\mathcal{M}$ in $\mathcal{H}_2$ and Bob measures $\mathcal{M}$ in $\mathcal{H}_4$. Each communicates the result (one bit of information) to the other participant. If the results are the same, go to step 3. If the results are different, Alice and Bob negate all bits in their possession, i.e., apply the unitary operation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

to each particle. *Step 3*: Alice applies the rotation

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

in $\mathcal{H}_1$ and Bob does the same in $\mathcal{H}_3$. *Step 4*: Alice performs the measurement $\mathcal{M}$ in $\mathcal{H}_1$ and Bob performs it in $\mathcal{H}_3$. Each communicates the result to the other. If the results agree then the states will have been swapped. If the results differ then Alice applies the unitary transformation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

to $\mathcal{H}_0$ and Bob does the same to $\mathcal{H}_5$ after which the states will have been swapped. A related process has been described by Vaidman [9].

It is interesting to compare the protocol to *quantum teleportation* [7], in which Alice and Bob initially share one maximally entangled pair and Alice is able to transfer an arbitrary state $|\xi\rangle$ to Bob by sending him only two classical bits of information. Thus, using the same resources as in our protocol, *viz.,* sharing two entangled pairs and each participant sending two bits to the other, we may alternatively swap the states $|\alpha\rangle$ and $|\beta\rangle$ by performing two teleportations (for the two directions of transfer.) However, the process above cannot be separated into two successive transfers. The remarkable feature of all these processes is that in the presence of shared entanglement an arbitrary state $|\xi\rangle$ may be transferred as a result of sending only a few bits of classical information despite the fact that $|\xi\rangle$ depends on two continuous parameters corresponding to an *infinite* amount of classical information.

The quantum controlled-NOT gate is not a universal gate. However, together with relatively trivial single-qubit operations it forms an adequate set of quantum gates, i.e., the set from which any quantum gate may be built [10]. Thus the conditional dynamics of the quantum controlled-NOT type would be in realistic technologies sufficient to construct any quantum information processing device. Universal two-qubit quantum gates based on similarly controlled dynamics are described in [11].

In the following, we outline two possible experimental realizations of the quantum controlled-NOT gate. We do not wish to suggest that these particular technologies are destined to yield practicable devices. They do, however, serve to illustrate physical considerations that would bear on the building of such devices in any technology.

The first technology is that of Ramsey atomic interferometry [12–15]; the second is based on the selective driving of optical resonances of two qubits undergoing a dipole-dipole interaction [16].

In the Ramsey atomic interferometry method the target qubit is an atom with two circular Rydberg states $|\epsilon_2\rangle$, where $\epsilon_2 = 0, 1$; the control qubit is the quantized electromagnetic field in a high-Q cavity C. The field in the cavity contains at most one photon of a particular mode so it can be viewed as a two state system with the vacuum state $|0\rangle$, and the one-photon state $|1\rangle$ as the basis. The cavity C is sandwiched between two auxiliary microwave cavities R_1 and R_2 in which classical microwave fields produce $\pi/2$ rotations of the Bloch vector of an atom passing through at a given speed,

$$|\epsilon_1\rangle_{\text{field}}|\epsilon_2\rangle_{\text{atom}} \longrightarrow |\epsilon_1\rangle_{\text{field}} \frac{1}{\sqrt{2}}[|\epsilon_2\rangle + (-1)^{\epsilon_2}e^{i\alpha}|1-\epsilon_2\rangle]_{\text{atom}}, \quad (9)$$

where the phase factor α is different for the two cavities R_1 and R_2. In the central cavity C, a dispersive interaction with the quantized field introduces phase shifts which depend on the state of the atom $|\epsilon_2\rangle$ and on the number of photons in the cavity $|\epsilon_1\rangle$. The interaction conserves the number of photons in the cavity:

$$|\epsilon_1\rangle_{\text{field}}|\epsilon_2\rangle_{\text{atom}} \longrightarrow \exp[i(-1)^{1-\epsilon_2}(\epsilon_1 + \epsilon_2)\theta]|\epsilon_1\rangle_{\text{field}}|\epsilon_2\rangle_{\text{atom}}\,, \quad (10)$$

where θ, the phase shift per photon, can be tuned to be π (θ depends on the time taken for the atom to cross C and the atom-field detuning).

The overall process can be viewed as a sequence: half flipping in R_1, phase shifts in C, and half flipping in R_2. Depending on the phase shifts, the second half flipping can either put the atom back into its initial state or flip it completely into the orthogonal state. The whole interferometer can be adjusted so that when the atom passes successively through the cavities, R_1, C, and R_2, the two qubits, i.e., the field and the atom, undergo the transformation

$$|\epsilon_1\rangle_{\text{field}}|\epsilon_2\rangle_{\text{atom}} \longrightarrow |\epsilon_1\rangle_{\text{field}}|\epsilon_1 \oplus \epsilon_2\rangle_{\text{atom}}\,. \quad (11)$$

The state of the field in C can also be transferred from (and to) an auxiliary Rydberg atom tuned to the resonant frequency of the cavity so that it undergoes a resonant rather than a dispersive interaction in C. This process allows the creation of a gate acting on two qubits of the same type, i.e., two Rydberg atoms rather than a field and an atom. Davidowich *et al.* [13] have shown how to use the Ramsey interferometry for quantum teleportation. Their experimental setup effectively contains conditional dynamics of the type we have been discussing, which has much wider application in quantum information processing than merely quantum teleportation. The practical realization of the quantum controlled-NOT gate can be achieved with a modification of the experiments as described in [13–15]. The typical resonant frequency would be of the order $\sim 2 \times 10^{10}$ Hz, the atom-field interaction time in the cavity $\sim 3 \times 10^{-5}$ s, and the cavity field lifetime can be made as long as ~ 0.5 s.

The most difficult part of the experimental realization is likely to be the preparation of a single atom. This is usually done by preparing an atomic beam with a very low probability of finding a single atom in the beam; consequently, finding two atoms in the beam is even less probable. From our point of view the drawback of this method is that it forces a trade-off between the probability that precisely one atom (as required) interacted with the field on a given run, and the reliability of the gate. Although our example has focused on microwave cavities, experimental realizations in the optical regime could also be considered [15].

Our second proposal for the implementation of the quantum controlled-NOT gate relies on the dipole-dipole interaction between two qubits. For the purpose of this model the qubits could be either magnetic dipoles, e.g., nuclear spins in external magnetic fields, or electric dipoles, e.g., single-electron quantum dots in static electric fields. Here we describe the model based on interacting quantum dots, but mathematically the two cases are isomorphic.

Two single-electron quantum dots separated by a distance R are embedded in a semiconductor. Let us consider the ground state and the first excited state of each dot as computation basis states $|0\rangle$ and $|1\rangle$. The first quantum dot, with resonant frequency ω_1, will act as the control qubit and the second one, with resonant frequency ω_2, as the target qubit. In the presence of an external static electric field, which can be turned on and off adiabatically in order to avoid transitions between the levels, the charge distribution in the ground state of each dot is shifted in the direction of the field while in the first excited state the charge distribution is shifted in the opposite direction (the *quantum-confined Stark effect*) [17]; see Fig. 1. In the simple model in which the state of the qubit is encoded by a single electron per quantum dot, we can choose coordinates in which the dipole moments in states $|0\rangle$ and $|1\rangle$ are $\pm d_i$, where $i = 1, 2$ refers to the control and target dots, respectively. For the sake of clarity, we are presenting the idea using a slightly simplified model. A more elaborate model would take into account holes in the valence band of the semiconductors. The state of a qubit would be determined by excitons of different energies.

The electric field from the electron in the first quantum dot may shift the energy levels in the second one (and vice versa), but to a good approximation it does not cause transitions. That is because the total Hamiltonian

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{V}_{12} \quad (12)$$

is dominated by a dipole-dipole interaction term $\hat{V}_{12}$ that is diagonal in the four-dimensional state space spanned by eigenstates $\{|\epsilon_1\rangle, |\epsilon_2\rangle\}$ of the free Hamiltonian $\hat{H}_1 + \hat{H}_2$, where ϵ_1 and ϵ_2 range over 0 and 1. Specifically,

$$(\hat{H}_1 + \hat{H}_2)|\epsilon_1\rangle|\epsilon_2\rangle = \hbar(\epsilon_1\omega_1 + \epsilon_2\omega_2)|\epsilon_1\rangle|\epsilon_2\rangle\,, \quad (13)$$

$$\hat{V}_{12}|\epsilon_1\rangle|\epsilon_2\rangle = (-1)^{\epsilon_1+\epsilon_2}\hbar\bar{\omega}|\epsilon_1\rangle|\epsilon_2\rangle\,, \quad (14)$$

where

$$\bar{\omega} = -\frac{d_1 d_2}{4\pi\epsilon_0 R^3}\,. \quad (15)$$

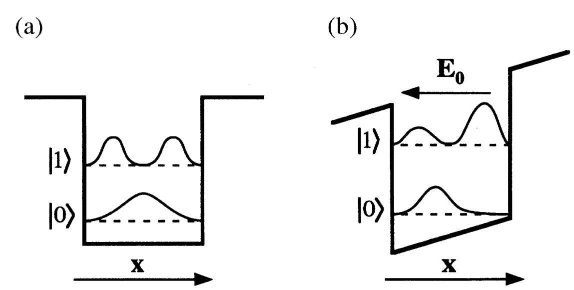

FIG. 1. Charge density in the quantum well in the direction **x** of the applied field. A dipole moment is induced when the electric field is turned on (b), but is zero without the electric field (a).

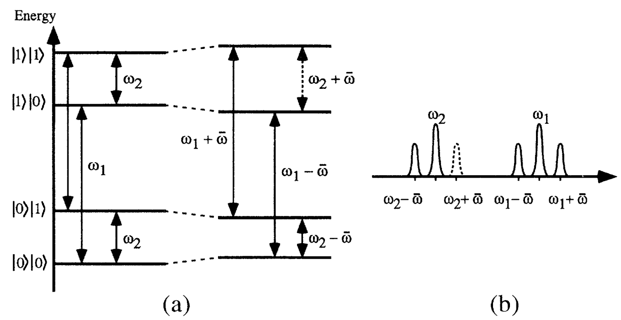

FIG. 2. (a) Energy levels of two quantum dots without and with the coupling induced by the presence of a static electric field $\mathbf{E}_0$. (b) Resonance spectrum of the two quantum dots. The dotted line shows the wavelength for which the two dots act as a controlled-NOT gate, with the first dot being the control qubit and the second the target qubit.

As shown in Fig. 2, it follows that due to the dipole-dipole interaction the resonant frequency for transitions between the states $|0\rangle$ and $|1\rangle$ of one dot *depends on the neighboring dot's state*. This is the conditional quantum dynamics we are seeking. The resonant frequency for the first dot becomes $\omega_1 \pm \bar{\omega}$ accordingly, as the second dot is in state $|0\rangle$ or $|1\rangle$, respectively. Similarly, the second dot's resonant frequency becomes $\omega_2 \pm \bar{\omega}$, depending on the state of the first dot. Thus a π pulse at frequency $\omega_2 + \bar{\omega}$ causes the transition $|0\rangle \rightarrow |1\rangle$ in the second dot if and only if the first dot is in state $|1\rangle$.

For such processes to be useful for quantum information processing, the decoherence time must be greater than the time scale of the optical interaction (see, for example, [18]). The decoherence time depends partly on the modification of the confining potential due to phononic excitations. There is also a quantum electrodynamic contribution due to coupling to the vacuum modes. For resonant frequencies in the infrared regime, this can be estimated at $\sim 10^{-6}$ s. Impurities and thermal vibration (phonons) can reduce the lifetime further to $\sim 10^{-9}$ s or even worse, but, in principle, their effects can be minimized by a more precise fabrication technology and by cooling the crystal. The optical interaction time scale can be approximated by the length of the π pulse ($\sim 10^{-9}$ s). The length of the pulse is not so much restricted by the current technology as by the requirement for the π pulse to be monochromatic and selective enough; this restricts the length of the pulse to being greater than the inverse of the pulse carrier frequency and the inverse of the dipole-dipole interaction coupling constant ($1/\bar{\omega} \sim 10^{-12}$ s in our model). This model may be more difficult to implement than the one based on the Ramsey atomic interferometry, but once it is implemented it is likely to allow for quantum gates to be integrated more easily into complex quantum circuits, as required for more general quantum information processing.

The authors thank B.G. Englert, S. Haroche, H.J. Kimble, H. Mabuchi, G. Mahler, J-M. Raimond, and H. Walther for discussions and comments. This work has been partially supported by the NIST Advanced Technology Program. A.B. gratefully acknowledges the financial support of the Berrow's Fund at Lincoln College (Oxford). Work by A.E. is supported by the Royal Society, London.

[1] D. Deutsch, Proc. R. Soc. London A **400**, 97 (1985).
[2] R. Feynman, Int. J. Theor. Phys. **21**, 467 (1982).
[3] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[4] D. Deutsch and R. Jozsa, Proc. R. Soc. London A **439**, 553 (1992); A. Berthiaume and G. Brassard, J. Mod. Opt. **41**, 2521 (1994); R. Josza, Proc. R. Soc. London A **435**, 563 (1991); D. Simon, in *Proceedings of the 35th Annual Symposium Foundations of Computer Science* (IEEE Press, New York, 1994), p. 116; E. Bernstein and U. Vazirani, in *Proceedings of the 25th ACM Symposium on Theory of Computation, 1993* (Association for Computing Machinery, New York, 1993), p. 11; P. W. Shor, in *Proceedings of the 35th Annual Symposium Foundations of Computer Science* (IEEE Press, New York, 1994).
[5] V. B. Braginsky, Yu. I. Vorontsov, and F. Ya. Khalili, Zh. Eksp. Theor. Fiz. **73**, 1340 (1977) [Sov. Phys. JETP **46**, 705 (1977)].
[6] S. L. Braunstein, A. Mann, and M. Revzen, Phys. Rev. Lett. **68**, 3259 (1992).
[7] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[8] R. Feynman, Opt. News **11**, 11 (1985).
[9] L. Vaidman, Phys. Rev. A **49**, 1473 (1994).
[10] A. Barenco, C. H. Bennett, R. Cleve, D. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. Smolin, and H. Weinfurter (to be published).
[11] D. Deutsch, Proc. R. Soc. London A **425**, 73 (1989); A. Barenco, Phys. Rev. Lett. (to be published); T. Sleator and H. Weinfurter, *ibid.* (to be published); D. Deutsch, A. Barenco, and A. Ekert, Proc. R. Soc. London A (to be published).
[12] N. F. Ramsey, *Molecular Beams* (Oxford University Press, New York, 1985).
[13] L. Davidovich, N. Zagury, M. Brune, J-M. Raimond, and S. Haroche, Phys. Rev. A **50**, R895 (1994); T. Sleator and H. Weinfurter (private communication).
[14] M. Brune, P. Nussenzveig, F. Schmidt-Kaler, F. Bernardot, A. Maali, J-M. Raimond, and S. Haroche, Phys. Rev. Lett. **72**, 3339 (1994).
[15] *Cavity QED*, Advances in Atomic, Molecular and Optical Physics, edited by P. Berman (Academic Press, New York, 1994), Suppl. 2.
[16] K. Obermayer, W. G. Teich, and G. Mahler, Phys. Rev. B **37**, 8096 (1988); W. G. Teich, K. Obermayer, and G. Mahler, *ibid.* **37**, 8111 (1988); W. G. Teich and G. Mahler, Phys. Rev. A **45**, 3300 (1992); see also S. Lloyd, Science **261**, 1569 (1993).
[17] D. A. B. Miller, D. S. Chemla, and S. Schmitt-Rink, Phys. Rev. B **33**, 6976 (1986).
[18] W. Unruh, Phys. Rev. A (to be published).