Network of Neural Oscillators for Retrieving Phase Information

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We propose a network of neural oscillators to retrieve given patterns in which the oscillators keep a fixed phase relationship with one another. In this description, the phase and the amplitude of the oscillators can be regarded as the timing and the strength of the neuronal spikes, respectively. Using the amplitudes for encoding, we enable the network to realize not only oscillatory states but also nonfiring states. In addition, it is shown that under suitable conditions the system has a Lyapunov function ensuring a stable retrieval process. Finally, the associative memory capability of the network is demonstrated numerically.

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Although the past decade has seen considerable advances in studies of neural networks, recent research seems to reveal the limitation of the current network models, composed of McCulloch-Pitts units or modifications of these. Provided that we will study the steady states of the network, these units are valid for the modeling of actual neurons as a first approximation. In fact, many fruitful results have been reported by using these units [1]. However, when it comes to treating the dynamical behavior of the network, these models may fail to capture the essence of the dynamics. This is because these models ignore much of the detailed behavior of real neurons, such as the timing of neuronal firing and internal dynamics [2–4]. In some cases, such behavior seems to play a significant role in neuronal systems. For example, several recent experiments suggest that the temporal coherence of neuronal activity, synchronization of pulse, may contribute to segmentation of visual scenes [5]. Central pattern generators (CGPs) provide another good example [6]. It is widely believed that a well-identified group of neurons (a CGP) controls rhythmic behavior of animals, i.e., locomotion, swimming, and so on. Since there are generally several distinct rhythmic patterns, it is obvious that the CGP can generate several firing patterns for which there are different phase relationships among the neuronal pulses.

To construct a theoretical model of the above systems, we need to describe the temporal features of neuronal activities, such as synchronization and phase locking. However, a McCulloch-Pitts description is based on the assumption that information is encoded only by the averaged activities of the neurons. Therefore, such a description is too crude to represent temporal features of the firing states naturally, for example, the relative phases of the spikes. Hence, we need to construct another model which provides a suitable framework to grasp such a temporal aspect of real neural networks. At the same time, it is required that such a model be simple enough to be mathematically tractable.

For this purpose, a model in which neuronal activities take the form of oscillators is an attractive candidate [7–9]. In this case, we may practically consider that such an oscillator represents a system of neurons which exhibits periodic behavior, instead of a single neuron. It is well known that in weakly coupled nonlinear oscillatory systems the complex original dynamics can be reduced to a simpler phase dynamics [10,11]. Using such a reduction technique, we can treat the timing of neuronal firings as the phase of the oscillators naturally. Moreover, it is desirable that the nonfiring state can also be expressed by the model. To express the nonfiring state, we will introduce an amplitude variable into our model phenomenologically. It is natural for the amplitude to be thought of as the strength of the pulse. Pursuing this idea, the nonfiring state can be represented by the zero amplitude of the oscillator. Because of the phenomenological nature of this amplitude, the theoretical relation between real systems and our model is less clear than in the case of the phase models. However, it is expected that the putative range of the applications of our model is extended by the description of the nonfiring state. Moreover, it seems that our model is useful to explore the role of temporal correlations among neurons in real systems and the possible applications of temporal coding in artificial neural networks. For the neuron model, therefore, we will adopt an oscillatory unit whose state is determined by both the phase and the amplitude.

Now we will construct a neural network model to retrieve phase information. For convenience of expression, we denote the state of the ith neuron by the complex variable W_i with amplitude r_i and phase ϕ_i. In this description, a nonfiring state of the neuron corresponds to $W_i = 0$, while a firing state corresponds to $W_i = \exp(i\Omega_i t)$ (Ω_i is the frequency of the ith firing neuron.) We consider a network of N neurons whose dynamics are governed by

$$\frac{dW_i}{dt} = v(W_i, \tilde{W}_i) + k\left(\sum_{j=1}^{N} C_{ij}W_j - W_i\right), \qquad (1)$$

where the real variable k represents the total coupling strength, the complex variable C_{ij} represents the effect of the interaction between the jth and ith neurons, and

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$\tilde{W}_i$ denotes the complex conjugate of W_i. The function $v(W_i, \tilde{W}_i)$ should be chosen so that the system can exhibit limited cycle behavior in the absence of the coupling term. In addition, we assume that the system (1) is invariant under uniform phase translation,

$$W_i \longrightarrow W_i \exp(i\phi_0), \qquad (2)$$

where ϕ_0 is an arbitrary real constant. This requirement arises from the fact that information is encoded not by the absolute time but by the relative time of neuronal spikes. Correspondingly, the relative phase relationships among the oscillators are relevant to encoding the information (not the absolute phase values). In this paper, for $v(W_i, \tilde{W}_i)$ we will examine two types of functions satisfying the above two requirements.

Here we define notation for later discussion. Let ξ_i^μ ($\mu = 1, \ldots, p$) be a set of patterns to be memorized, where p is the total number of the patterns. We should remark that because of the invariance (2), all patterns generated by the uniform phase translation $\xi_i^\mu \exp(i\phi_0)$ represent the same pattern as ξ_i^μ. We also define p *overlaps* M_μ as the projections of the current state on the p embedded patterns,

$$M_\mu = \frac{1}{N}\left|\sum_{j=1}^{N} \tilde{\xi}_i^\mu W_j\right|. \qquad (3)$$

At first, we consider the situation that all neurons are in the firing state. Stored patterns can then be characterized by the parameters θ_i^μ [$\xi_i^\mu = \exp(i\theta_i^\mu)$]. Because of (2), the effective number of degrees of freedom of these parameters is $N - 1$ for one pattern. As a simple choice, we consider the following dynamics of the first model:

$$\frac{dW_i}{dt} = (1 + i\Omega_i)W_i - (1 + ic)|W_i|^2 W_i + k\left(\sum_{j=1}^{N} C_{ij}W_j - W_i\right), \qquad (4)$$

where Ω_i can be considered as natural frequencies of neuronal spikes in the absence of external inputs and c is a real parameter. The system (4) without the coupling terms is often referred to as the Stuart-Landau equation. In general, it is well known that the Stuart-Landau equation is derived from a general ordinary differential equation near a Hopf bifurcation point [10]. Therefore, this choice seems to be natural for oscillatory systems. For simplicity, we assume that $c = 0$ and $\Omega_i = \Omega$. Under this assumption, we can eliminate Ω by the transformation $W_i \longrightarrow W_i \exp(i\Omega t)$. After simple calculations, therefore, (4) reduces to

$$\frac{dW_i}{dt} = W_i - |W_i|^2 W_i + k\left(\sum_{j=1}^{N} C_{ij}W_j - W_i\right). \qquad (5)$$

Next we will show how given patterns are embedded in the network of oscillators, that is, how to determine the synaptic efficacies. At first, let us consider the condition that Eq. (5) has the solution $W_i = \xi_i^\mu$. Substituting $W_i = \xi_i^\mu$ into (5) and using $|\xi_i^\mu| = |\exp(i\theta_i^\mu)| = 1$, we obtain $\sum_{j=1}^N C_{ij}\xi_j^\mu = \xi_i^\mu$. This condition can be rewritten in matrix form as $\mathbf{CP} = \mathbf{P}$ where the $N \times P$ matrix $\mathbf{P}$ is defined by $P_{ij} = \xi_i^j$. Provided that the total number p of the memorized patterns is smaller than N, the synaptic efficacies satisfying this condition are then given by $\mathbf{C} = \mathbf{PP}^\dagger + \mathbf{B}(\mathbf{I} - \mathbf{PP}^\dagger)$ where $\mathbf{P}^\dagger$ denotes the pseudoinverse of $\mathbf{P}$, $\mathbf{I}$ denotes the unit matrix, and $\mathbf{B}$ is an arbitrary matrix [12,13]. If the patterns are linearly independent, the pseudoinverse matrix $\mathbf{P}^\dagger$ is given by $\mathbf{P}^\dagger = 1/N(1/N\tilde{\mathbf{P}}^T\mathbf{P})^{-1}\tilde{\mathbf{P}}^T$, where $\tilde{\mathbf{P}}$ and $\mathbf{P}^T$ denote the complex conjugate and the transpose of $\mathbf{P}$, respectively. Moreover, assuming $\mathbf{B} = \mathbf{O}$ for simplicity, we obtain

$$C_{ij} = \frac{1}{N}\sum_{k=1}^{p}\sum_{l=1}^{p}(A^{-1})_{kl}\xi_i^k\tilde{\xi}_j^l, \quad A_{ij} = \frac{1}{N}\sum_{k=1}^{N}\xi_k^i\tilde{\xi}_k^j, \qquad (6)$$

with $\xi_i^\mu = \exp(i\theta_i^\mu)$. It is easily found that the synaptic matrix (6) is Hermitian, that is, $C_{ij} = \tilde{C}_{ji}$. Note that the synaptic prescription (6) recovers the generalized Hebbian rule $C_{ij} = (1/N)\sum_{k=1}^{p}\xi_i^\mu\tilde{\xi}_j^\mu$ if the patterns are orthonormal.

There are two trivial cases in this model. For $p = 1$, the synaptic efficacies are given by $C_{ij} = (1/N)\exp(i\theta_i^1 - i\theta_j^1)$. In this case, applying the transformation $W_i \exp(-i\theta_i) \longrightarrow W_i$, the dynamics of (5) reduce to

$$\frac{dW_i}{dt} = W_i - |W_i|^2 W_i + k\left(\frac{1}{N}\sum_{j=1}^{N} W_j - W_i\right). \qquad (7)$$

This equation has recently been studied by several authors [14,15]. This situation corresponds to the Mattis state in spin glass models. On the other hand, for $p = N$ we obtain $C_{ij} = \delta_{ij}$. This implies that the couplings between the oscillators vanish and the dynamics of the network reduces to that of N independent oscillators.

We will show that the system (4) reduces to a simpler set of phase oscillators under the conditions that the coupling is weak and $c = 0$. In the limit of weak coupling $k \sim O(\epsilon)$, the deviation of the amplitude r_i from $r = 1.0$ may be neglected at first order. Inserting $W_i = r_i \exp(i\phi_i)$ into (5), this approximation immediately yields

$$\frac{d\phi_i}{dt} = \Omega_i + k\sum_j |C_{ij}|\sin(\phi_j - \phi_i + \alpha_{ij}), \qquad (8)$$

where α_{ij} are defined by $C_{ij} = |C_{ij}|\exp(i\alpha_{ij})$. Since the synaptic matrix is Hermitian, we get $|C_{ij}| = |C_{ij}|$ and $\alpha_{ij} = -\alpha_{ji}$. Note that $\alpha_{ij} = -\alpha_{ji}$ leads $\alpha_{ii} = 0$. As a result, self-coupling terms do not contribute to the dynamics of the network.

In the previous model (4), all neurons are assumed to exhibit periodic firing states. However, such an assumption is not realistic because in a real system, for any given patterns, some neurons will be in a resting state.

In the second model, we consider another type of dynamics where the amplitude plays a crucial role in representing the nonfiring state. To realize a stable nonfiring state in retrieval patterns, it is desirable that the single neuron is able to exhibit both a periodic firing state and a nonfiring state. To satisfy this requirement, we choose $v(W_i, \tilde{W}_i)$ so that the dynamics of a single neuron can have both a limited cycle $W_i = \exp(i\Omega_i t)$ and a stationary fixed point $W_i = 0$. As a simple choice, we employ $v(W_i, \tilde{W}_i) = (-1 + i\Omega_i)W_i + 4|W_i|^2 W_i - 3|W_i|^4 W_i$. As is the case of the first model, assuming that the natural frequencies Ω_i are identical to Ω, we can eliminate Ω. Consequently, the dynamics of the network are described by the following:

$$\frac{dW_i}{dt} = - W_i + 4|W_i|^2 W_i - 3|W_i|^4 W_i + k\left(\sum_{j=1}^{N} C_{ij} W_j - W_i\right). \tag{9}$$

We next address the question of how to make the synaptic connections for patterns which include the possibility of nonfiring states. The patterns are then defined by

$$\xi_i^\mu = \begin{cases} \exp(i\theta_i^\mu) & \text{for firing state}, \\ 0 & \text{for nonfiring state}. \end{cases} \tag{10}$$

Putting the above patterns into (9) and using $|\xi_i^\mu| = 1$ or 0, we obtain the same condition for the synaptic connections $\mathbf{CP} = \mathbf{P}$ as in the first model. Therefore, if $p < N$ and the patterns are linearly independent, we can apply the same procedure as in the first model. As a result, the synaptic matrix is given by the same prescription (6). Note that the condition $C_{ij} = \tilde{C}_{ji}^T$ is kept even if the memorized patterns include some nonfiring states.

Above, it was shown that we can set the connections so as to make the patterns to be memorized ξ_i^μ the solutions of the dynamical equation (1). To recall the embedded patterns from noisy ones, however, it is required that such solutions are the attractors of the dynamics. The existence of such asymptotic behavior is guaranteed by the existence of a Lyapunov function. Here, we will show that in our models a Lyapunov function exists if the synaptic matrix is Hermitian and $\Omega_i = \Omega$. The models which we have discussed so far can be written in the form

$$\frac{dW_i}{dt} = v(W_i, \tilde{W}_i) + k\left(\sum_{j=1}^{N} C_{ij} W_j - W_i\right), \tag{11}$$

with

$$v(W_i, \tilde{W}_i) = \begin{cases} W_i - |W_i|^2 W_i, \\ -W_i + 4|W_i|^2 W_i - 3|W_i|^4 W_i. \end{cases} \tag{12}$$

Let us introduce a function $V(W_i, \tilde{W}_i)$ by the following definition:

$$v(W_i, \tilde{W}_i) = -\frac{\partial V(W_i, \tilde{W}_i)}{\partial \tilde{W}_i}$$

and

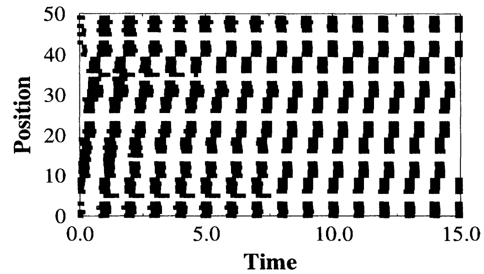

FIG. 1. Typical temporal pattern of retrieval process in the second model. Black horizontal bars show active phases which are defined by $\mathrm{Re}W_i > 0.5$. The network succeeds in retrieving one of eight memorized patterns. Note that the memorized pattern includes nonfiring states.

$$\tilde{v}(W_i, \tilde{W}_i) = -\frac{\partial V(W_i, \tilde{W}_i)}{\partial W_i}, \tag{13}$$

where we have regarded W_i and $\tilde{W}_i$ as independent variables. For example, the corresponding functions $V(W_i, \tilde{W}_i)$ for (12) are given by

$$V(W_i, \tilde{W}_i) = \begin{cases} -|W_i|^2 + \frac{1}{2}|W_i|^4, \\ |W_i|^2 - 2|W_i|^4 + |W_i|^6. \end{cases} \tag{14}$$

Provided that such a function $V(W_i, \tilde{W}_i)$ exists, the Lyapunov function is given by

$$L(W_i, \tilde{W}_i) = \sum_{i=1}^{N} V(W_i, \tilde{W}_i) - \frac{k}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} (C_{ij} \tilde{W}_i W_j + \tilde{C}_{ij} W_i \tilde{W}_j) + k \sum_{i=1}^{N} |W_i|^2. \tag{15}$$

It is easily proved that L only decreases under the dynamics of (11). Using the function $V(W_i, \tilde{W}_i)$, we can rewrite the model (11) in the form $dW_i/dt = -\partial L/\partial \tilde{W}_i$ and $d\tilde{W}_i/dt = -\partial L/\partial W_i$. From these relations, we immediately get

$$\frac{dL}{dt} = \sum_i^N \left(\frac{\partial L}{\partial W_i}\frac{dW_i}{dt} + \frac{\partial L}{\partial \tilde{W}_i}\frac{d\tilde{W}_i}{dt}\right) = -\sum_i^N \left| v(W_i, \tilde{W}_i) + k\left(\sum_{j=1}^{N} C_{ij} W_j - W_i\right)\right|^2 \leq 0. \tag{16}$$

Therefore, L can only decrease as a function of time. When the network converges to the memorized state $W_i = \xi_i^\mu$, L does not vary with time. Note that L is invariant under the uniform phase translation (2).

To confirm the ability of the network, we carried out some numerical simulations. Among these simulations, we present one typical result. In this simulation, we used a network of 50 oscillators whose dynamics are governed by Eq. (9). The reason for using the second model is that

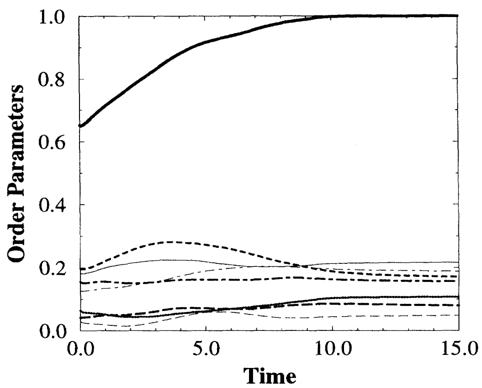

FIG. 2. Time development of eight overlaps in the same simulation as Fig. 1. The solid line shows the overlap M_1 concerning the retrieval pattern ξ^1.

we want to demonstrate the ability of the network to retrieve phase patterns which include nonfiring states. In all, eight patterns were stored by means of the synaptic prescription (6). Figure 1 shows one typical temporal pattern of the retrieval process in which the parameter values are $k = 1.0, \Omega = 2\pi$. Although Ω make no contribution to the dynamics of the network, to express the phase relationships visually, the pattern in Fig. 1 is illustrated in the rotating frame. In this context, the black horizontal bars represent active phases defined by $\mathrm{Re}W_i > 0.5$. The network was initially given the noisy pattern ξ_i^1. The first pattern was $\xi^1 = (1,1,1,1,0,0,1,1,1,1,e^{i(2\pi/5)},e^{i(2\pi/5)},e^{i(2\pi/5)},e^{i(2\pi/5)},0,0,e^{i(2\pi/5)},e^{i(2\pi/5)},e^{i(2\pi/5)},e^{i(2\pi/5)},\ldots,e^{i(8\pi/5)},e^{i(8\pi/5)},e^{i(8\pi/5)},e^{i(8\pi/5)},0,0,e^{i(8\pi/5)},e^{i(8\pi/5)},e^{i(8\pi/5)},e^{i(8\pi/5)})$. The remaining seven patterns ξ_i^μ ($p = 2-8$) were generated at random. That is, taking the form $\xi_i^\mu = A_i^\mu \exp(i\theta_i^\mu)$, θ_i^μ were chosen at random from a uniform distribution between 0 and 2π, and A_i^μ were independent random variables obeying the probability distribution $P(A_i^\mu) = \frac{1}{5}\delta(A_i^\mu - 1) + \frac{4}{5}\delta(A_i^\mu)$. Obviously, both the amplitudes and the phases were corrected dynamically. In the same simulation, time development of overlaps is shown in Fig. 2. It is clearly found that the network succeeds in retrieving one of eight memorized patterns.

We would like to make some comments here before concluding. First, the synaptic efficacies C_{ij} were regarded as complex numbers. This may be explained naturally by the fact that the neurons are coupled via more than one component [16]. Secondly, even if the natural frequencies Ω_i have a nontrivial distribution, it is expected that for $k > k_c$ (a certain critical value) the retrieval of the patterns will be achieved by the network. Finally, it is well known that the storage capacity of an oscillator network constructed by using the Hebbian rule is given by $\alpha_c = P/N = 0.0377$ [17–19]. Therefore, using the generalized Hebbian rule, we expect that our models work well when $\alpha < \alpha_c$.

In conclusion, we proposed a network of oscillators for the retrieval of phase information. We showed that this network has the ability to retrieve given patterns in which the oscillators keep a fixed phase relationship with one another. We would like to emphasize that in the case of the second model, the network can retrieve patterns which include nonfiring states. Furthermore, it is shown that under suitable conditions the system has a Lyapunov function ensuring a stable retrieval process. Using numerical simulations, we confirmed the good performance of our models. Consequently, we believe that the proposed models serve as a convenient starting point for the study of oscillatory neuronal systems.

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