# Universal Fluctuations of Chern Integers

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Chern integers describing quantized transport change (typically by ±1) when the energy levels or bands with which they are associated become degenerate. We give a statistical treatment of these degeneracies and the consequent fluctuations of the Chern integers. The density of degeneracies is calculated exactly for a parametrization of the Gaussian unitary ensemble of matrices, and we present numerical results indicating that the "gas" of degeneracies has a "charge neutrality" property. The results apply to a broad class of complex systems after rescaling the parameter space.

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The Chern integer is a topological invariant which can characterize quantized transport without dissipation. It occurs in the analysis of the quantized Hall effect for electrons in a crystal lattice without disorder [1], and in models for transport with a time dependent periodic potential [2]. It also arises in more general models for quantized conductance [3], in which the potential need not be periodic. In all of these systems the Hamiltonian can be represented by a Hermitian operator which is a periodic function of two parameters, either Bloch wave vectors or magnetic fluxes threaded through holes in the sample; we denote these parameters by $X_1$ and $X_2$, and assume that they are scaled so that both periods are $2\pi$. There is a Chern integer $N_n$ associated with each eigenvalue $E_n$ of the operator. In some systems the Chern integers may be large and impossible to calculate analytically. An example is illustrated in Fig. 1; the system is a chaotic quantum billiard pierced by three magnetic fluxes, $\hbar X_i/e$, $i = 1,2,3$. Chern integers are associated with each energy level, describing quantized transfer of charge around the second flux in response to increasing the first flux by one quantum. This Letter is the first of a pair of publications which characterize the Chern integers statistically, by considering the effect of varying a third parameter $X_3$. It is not essential that the Hamiltonian be periodic in $X_3$; this parameter could describe a change in the shape of the boundary or an externally applied electric field instead of a flux.

Pairs of eigenvalues typically degenerate at isolated points in the space of the three parameters $(X_1, X_2, X_3)$. The Chern integers generically change by $\pm 1$ at values of $X_3$ for which there is a degeneracy [4]; a sign can be attached to each degeneracy, positive if the Chern number of the lower degenerating level increases when $X_3$ increases past the point of degeneracy, negative otherwise. Reference [5] lists some earlier papers which have investigated the changes of Chern numbers at degeneracies. If the positions and signs of degeneracies were randomly distributed in both parameter space and over energy level labels, the Chern integers would perform a random walk as $X_3$ is varied. We will therefore use the following relation to characterize the change in the Chern integer of the $n$th level, $\Delta N_n$, caused by a change $\Delta X_3$ of the third parameter:

$$\langle \Delta N_n^2 \rangle = 2\,\chi(\Delta X_3)\,\mathcal{A}\,\mathcal{D}\,\Delta X_3\,. \tag{1}$$

Here $\mathcal{D}$ is the density of degeneracies in parameter space between a given level and one of its neighbors, $\mathcal{A} = 4\pi^2$ is the area of the $(X_1, X_2)$ torus, and $\chi(\Delta X_3)$ is a function which contains information about the correlations between positions of degeneracies: If they were randomly distributed, we would have $\chi = 1$.

There is very strong evidence that random matrix ensembles provide a good description of complex systems without symmetries or constants of motion: Examples include many-body systems such as nuclei [6], and systems with few degrees of freedom but with chaotic classical dynamics [7], such as our billiard example. The agreement with random matrix theory is strongest for statistics involving energy levels in a small range; statistics of degeneracies provide a good example. In order to calculate the density of degeneracies in parameter space we require a parameter dependent random matrix model. A very natural model for our purposes is the following [8]:

$$\hat{H}(\mathbf{X}) = \sum_{i=1}^{3} \cos X_i \hat{H}_{2i-1} + \sin X_i \hat{H}_{2i}\,, \tag{2}$$

in which $\hat{H}_i$ are six independent realizations of the Gaussian unitary ensemble (GUE). This model has several con-

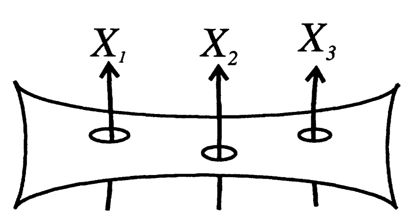


FIG. 1. Our results model dissipationless charge transport in a chaotic quantum billiard pierced by three fluxes.

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venient features: Its statistical properties are homogeneous in the parameter space and $d\hat{H}/d\mathbf{X}$ is independent of $\hat{H}$, while both are drawn from Gaussian unitary ensembles. Also, the unitary invariance property implies that the statistics of $d\hat{H}/d\mathbf{X}$ are unchanged when we transform to the eigenbasis of $\hat{H}$. The real and imaginary parts of $\hat{H}_i$ are symmetric and antisymmetric matrices with independently Gaussian distributed elements; their variances are

$$\langle(\mathrm{Re}H_{ij})^2\rangle = 1 + \delta_{ij}, \quad \langle(\mathrm{Im}H_{ij})^2\rangle = 1 - \delta_{ij}. \tag{3}$$

The statistical properties of (2) are expected to be the same as those of a typical quantum system under an appropriate transformation of the parameter space and the level density [8,9]. The sensitivity of a system to a perturbation parameter $X_i$ is conveniently described in terms of statistics of the matrix elements of $\partial\hat{H}/\partial X_i$ in the basis formed by the eigenstates $|\phi_n\rangle$ of $\hat{H}$: We will use the notation $(\partial H/\partial X_i)_{nm} = \langle\phi_m|\partial\hat{H}/\partial X_i|\phi_n\rangle$ for these matrix elements. They will be characterized by correlation coefficients for matrix elements between states close to a given energy $E$:

$$C_{ij}(E) = \left\langle \left(\frac{\partial H}{\partial X_i}\right)^*_{nm} \left(\frac{\partial H}{\partial X_j}\right)_{nm} \right\rangle_{E_n \sim E_m \sim E}, \tag{4}$$

and we will use the symbol $\tilde{C}$ to denote the $3 \times 3$ matrix with coefficients $C_{ij}$. A precise definition of the average over the matrix elements denoted by the angle brackets, and its relationship to classical correlation functions, is discussed in [10]. In the case of our parametrized GUE model, these correlation coefficients are $C'_{ij} = 2\delta_{ij}$: this follows from Eqs. (2) and (3) and the invariance of the ensemble under unitary transformations. A linear transformation of the parameter space can be used to reduce the matrix $\tilde{C}$ to the diagonal form $\tilde{C}' = 2\tilde{I}$; the statistics of the degeneracies are then expected to appear (locally) the same as those of the GUE system. Correlation functions of matrix elements such as (4) contain information about both the Ohmic or dissipative conductances of our billiard system and the mean value of the Chern integers; we will discuss these issues in our later paper.

First we will calculate the density of degeneracies. The approach is to select an arbitrary point $\mathbf{X}$, and calculate the distance $R$ to the nearest degeneracy between levels $n$ and $n + 1$, using degenerate perturbation theory (this approximation is only accurate if $R$ is small). We use the resulting expression to calculate the probability that a degeneracy exists at some small distance $R$ from our randomly chosen point. This probability must be $P[R]\,dR = 4\pi R^2 \mathcal{D}\, dR$, where $\mathcal{D}$ is the mean density of degeneracies between the two levels; having calculated $P[R]$, we can deduce $\mathcal{D}$. A similar approach has previously been applied to the simpler case of the Gaussian orthogonal ensemble [11].

According to two-state degenerate perturbation theory, the degeneracy occurs when a discriminant vanishes, at a displacement $\delta\mathbf{X}$ from our reference point. The components $\delta X_i$ of this displacement are given by the following three linear equations:

$$\sum_{i=1}^{3} \partial_i H_j \, \delta X_i = -\Delta\, \delta_{1j}, \qquad j = 1,2,3, \tag{5}$$

where $\Delta = E_{n+1} - E_n$, and

$$\partial_i H_1 = \left(\frac{\partial H}{\partial X_i}\right)_{n,n} - \left(\frac{\partial H}{\partial X_i}\right)_{n+1,n+1},$$
$$\partial_i H_2 = \mathrm{Re}\left(\frac{\partial H}{\partial X_i}\right)_{n,n+1}, \qquad \partial_i H_3 = \mathrm{Im}\left(\frac{\partial H}{\partial X_i}\right)_{n,n+1} \tag{6}$$

(all evaluated at $\mathbf{X}$). The distance from the reference point to the degeneracy $R = |\delta\mathbf{X}|$ is proportional to $\Delta$: We write $R = \Delta f$ where

$$f = \sqrt{w_1^2 + w_2^2 + w_3^2}/w,$$
$$w = \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\, \partial_i H_1\, \partial_j H_2\, \partial_k H_3,$$
$$w_n = \sum_{\substack{i,j=1 \\ i,j\neq n}}^{3} \varepsilon_{ij} \partial_i H_2 \partial_j H_3, \tag{7}$$

and $\varepsilon_{ij\ldots}$ is the Levi-Civita symbol. Since $\Delta$ depends on the matrix elements of the Hamiltonian and $f$ on the matrix elements of its derivatives, these quantities are statistically independent in our model. The probability $P[R]$ that the nearest degeneracy exists at a small distance $R$ can then be written

$$P[R] = \int_0^\infty df \int_0^\infty d\Delta\, P[f]P[\Delta]\delta(R - f\Delta), \tag{8}$$

where $P[f]$ is the probability distribution for $f$, and $P[\Delta]$ is the distribution of neighboring energy level separations. When $R$ is small, the Dirac delta function only supports small values of $\Delta$, the probability distribution of which is well known [12]: $P[\Delta]\,d\Delta = \frac{1}{3}\pi^2\rho^3\Delta^2 d\Delta$, where $\rho$ is the density of states. This gives

$$P[R] = \tfrac{1}{3}\,\pi^2\rho^3\langle f^{-3}\rangle R^2. \tag{9}$$

After a lengthy calculation, the essentials of which we will explain in our subsequent paper, we find $\langle f^{-3}\rangle = 8\sqrt{8/\pi}$.

Equating (9) with $4\pi R^2 \mathcal{D}\, dR$ then gives the density of degeneracies directly. For a general system, it is necessary to scale the density of degeneracies by the Jacobian of the coordinate transformation which reduces the correlation matrix $\tilde{C}$ to the diagonal form $\tilde{C}' = 2\tilde{I}$ of the GUE model: The density of degeneracies is therefore

$$\mathcal{D} = \frac{2}{3}\sqrt{\pi}\rho^3\sqrt{\det(\tilde{C})} = \frac{2}{3}\sqrt{\pi}\rho^3\sigma^3, \tag{10}$$

where the second equality defines $\sigma$, which is a measure of the typical magnitude of the matrix elements $(\partial H/\partial X_i)_{nm}$ [for our GUE model, it follows from (2), (3), and (4) that $\sigma = \sqrt{2}$). We tested this result by counting the total number of degeneracies $\Sigma$, for all energy levels, in the $(2\pi)^3$ parameter space volume of our GUE model. Using the Wigner semicircle law [12] for the density of states, (10) implies that $\Sigma = 512M^{5/2}/135\sqrt{3}\pi$, where $M$ is the dimension of the GUE matrices; this prediction is compared with numerical experiments in Table I.

We now consider how to calculate the function $\chi$ appearing in Eq. (1). Consider the effect of a change in the third parameter $\Delta X_3$ on the Chern integers defined on the $(X_1, X_2)$ torus. The Chern integer of the $n$th level changes by $\pm 1$ at degeneracies, depending on the sign of the degeneracy and on whether the degeneracy is with the level above or below: The total change in the Chern integer is $\Delta N_n = \mathcal{N}_n^+ - \mathcal{N}_n^- - \mathcal{N}_{n-1}^+ + \mathcal{N}_{n-1}^-$, where $\mathcal{N}_n^s$ is the number of degeneracies of sign $s$ occurring between levels $n$ and $n+1$ in the volume $V(\Delta X_3)$ swept out as $X_3$ is varied. The $\mathcal{N}_n^s$ can be obtained by integrating a density of degeneracies $D_n^s(\mathbf{X})$:

$$\mathcal{N}_n^s = \int_{V(\Delta X_3)} d\mathbf{X}\, D_n^s(\mathbf{X}),$$
$$D_n^s(\mathbf{X}) = \sum_j \delta(\mathbf{X} - \mathbf{X}_j(n,s)). \tag{11}$$

Here $\mathbf{X}_j(n,s)$ are the positions of the degeneracies between levels $n$ and $n+1$ of sign $s$. The random function $D_n^s(\mathbf{X})$ can be characterized by its mean value and correlation function. We will assume that the density of states is slowly varying on the scale of the mean level separation, implying that the dependence of these statistics on energy can be neglected in the following arguments. Also, because $s$ depends on the orientation of the parameter space, which is arbitrary in our GUE model, there is symmetry with respect to the signs. We therefore write $\langle D_n^s(\mathbf{X})\rangle = \mathcal{D}/2$, where $\mathcal{D}$ is the mean density irrespective of sign, and define a set of correlation functions $C_{n-n'}^{s\times s'}(\mathbf{X})$ as follows:

$$\langle D_n^s(\mathbf{X})\, D_{n'}^{s'}(\mathbf{X}')\rangle = \tfrac{1}{2}\,\mathcal{D}\,\delta_{nn'}\,\delta_{ss'}\,\delta(\mathbf{X} - \mathbf{X}') + C_{n-n'}^{s\times s'}(\mathbf{X} - \mathbf{X}'). \tag{12}$$

Using considerations of homogeneity to equate terms, we have

$$\langle \Delta N_n\rangle = 0, \quad \langle \Delta N_n^2\rangle = 4\langle(\mathcal{N}_n^+)^2\rangle - 4\langle \mathcal{N}_n^+\mathcal{N}_n^-\rangle - 4\langle \mathcal{N}_n^+\mathcal{N}_{n-1}^+\rangle + 4\langle \mathcal{N}_n^+\mathcal{N}_{n-1}^-\rangle. \tag{13}$$

Using the correlation functions defined in (12), we find

$$\langle \Delta N_n^2\rangle = 2\mathcal{A}\mathcal{D}\Delta X_3 + 4\int_{V(\Delta X_3)} dV \int_{V(\Delta X_3)} dV' [C_0^+(\mathbf{X} - \mathbf{X}') - C_0^-(\mathbf{X} - \mathbf{X}') - C_1^+(\mathbf{X} - \mathbf{X}') + C_1^-(\mathbf{X} - \mathbf{X}')], \tag{14}$$

which can be written in the form of Eq. (1).

We now discuss the limiting forms of the function $\chi(\Delta X_3)$. In the following considerations we will ignore the fact that (2) is periodic in $X_3$, and assume that the correlation functions $C_n^\pm(\mathbf{X})$ decay to zero as $|\mathbf{X}| \to \infty$. In the limit $\Delta X_3 \to 0$, the double integral in (14) makes a contribution which is $O(\Delta X_3^2)$, implying that

$$\lim_{\Delta X_3 \to 0} \chi(\Delta X_3) = 1, \tag{15}$$

which is equivalent to regarding the degeneracies as uncorrelated. In Fig. 2 we display numerical results showing the diffusion of the Chern integers, compared with both (14) and the linear approximation (15). If the correlation functions decay sufficiently rapidly, (14)

TABLE I. Total numbers of degeneracies present in the $8\pi^3$ volume of parameter space. The data are for one realization of the GUE model, with different values of the matrix dimension $M$.

| $M$ | $\Sigma_{\text{numerical}}$ | $\Sigma_{\text{theory}}$ |
|---|---|---|
| 5 | 64 | 69 |
| 7 | 160 | 160 |
| 10 | 436 | 391 |
| 15 | 1082 | 1077 |
| 20 | 2214 | 2210 |

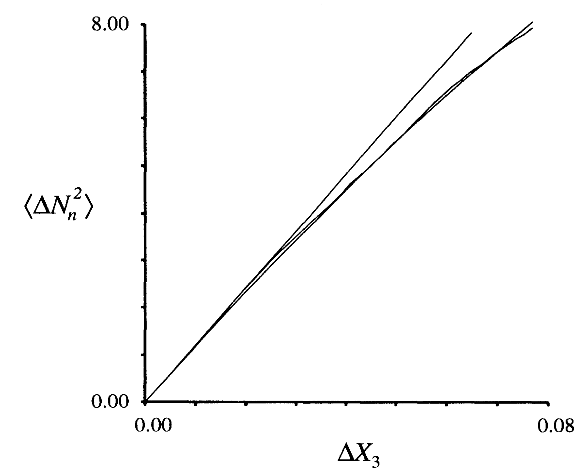


FIG. 2. Diffusion of Chern integers for small $\Delta X_3$. The straight line is for uncorrelated degeneracies ($\chi = 1$), the smooth curve was calculated from (14), using numerically determined correlation functions, and the third line is $\langle \Delta N^2\rangle$ averaged over many realizations. The data are for levels close to the center of the spectrum of $40 \times 40$ GUE matrices.

implies that $\langle \Delta N_n^2 \rangle$ grows linearly as $\Delta X_3 \to \infty$, unless

$$\mathcal{D} + 2 \int dV [C_0^+(\mathbf{X}) - C_0^-(\mathbf{X}) - C_1^+(\mathbf{X}) + C_1^-(\mathbf{X})] = 0 \tag{16}$$

(here the integral is over all space). If the sum rule (16) is satisfied, we expect that $\langle \Delta N_n^2 \rangle$ approaches a constant as $\Delta X_3$ increases. The distribution of degeneracies can be thought of as a gas of particles of different types labeled by integers $n$ and charges $s = \pm 1$. We will present numerical evidence that (16) holds, implying that "charge fluctuations" in the "gas" of degeneracies are perfectly screened.

After transforming the parameter space to reduce the correlation matrix $\tilde{C}$ to a multiple of the identity, the correlation functions defined in (12) are "universal" functions of the dimensionless variable $x = \sigma \rho |\mathbf{X}|$. It follows that the function $\chi(\Delta X_3)$ can be written in terms of a universal scaling function $f(x)$: $\chi(\Delta X_3) = f(\rho \sigma \Delta X_3)$. Equation (15) implies that $f(0) = 1$, and (16) implies that $\chi(\Delta X_3)\Delta X_3$ approaches a constant as $\Delta X_3 \to \infty$, or $f(x)x \to \mu$ as $x \to \infty$, where $\mu$ is a universal constant. Also, if $\langle \Delta N_n^2 \rangle$ saturates when $\rho \sigma \Delta X_3 \gg 1$, we can relate the variance of the Chern integers to the saturation value of this statistic: $\mathrm{var}(N_n) = \frac{1}{2} \langle \Delta N_n^2 \rangle_{\mathrm{sat}}$. Our hypothesis (16) therefore implies that

$$\mathrm{var}(N_n) = \tfrac{1}{2} \langle \Delta N_n^2 \rangle_{\mathrm{sat}} = \frac{\mu \mathcal{A} \mathcal{D}}{\rho \sigma} = \tfrac{2}{3} \sqrt{\pi} \mu \mathcal{A} \rho^2 \sigma^2. \tag{17}$$

We computed the Chern integers for our parametrized GUE model at $X_3 = 0$, and extended the results through the parameter space by identifying the degeneracies and their signs. Our results, for matrices of dimension $M$ up to 50, did not give good enough statistics to test (16) or (17) directly; instead we calculated the statistic

$$S = \sum_{n=1}^{M} \langle N_n^2 \rangle = \frac{16\pi^2}{3} \sqrt{\pi} \mu \int dE\, \rho^3(E) = \tfrac{2}{3} \sqrt{\pi} \mu M^2 \tag{18}$$

for our parametrized GUE model (we used the "semicircle law" [12] for the density of states at large $M$ to estimate the integral). We present a plot of $S$ against $M$ in Fig. 3, which confirms the prediction that $S \sim M^2$, and gives the estimate $\mu \approx 0.2$.

To summarize, we have calculated the density of degeneracies exactly, and we have presented numerical results indicating that charge fluctuations are perfectly screened; we used these results to characterize the fluctuations of the Chern integers. In our subsequent paper, we will describe a variety of analytical results on sums of Chern integers, which are relevant to systems modeled by independent fermions.

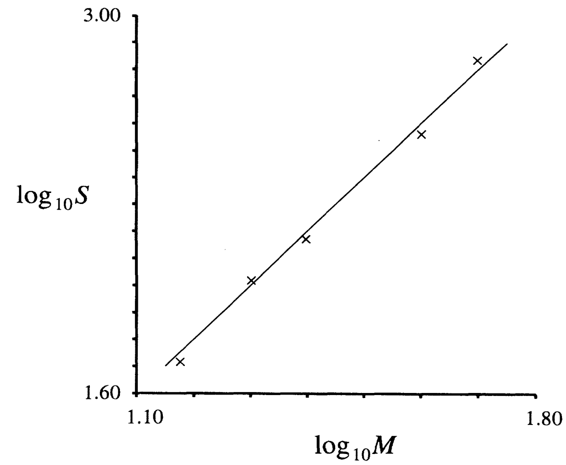


FIG. 3. Numerical evidence for (18): The gradient of the line is 2.

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