Trajectory (Phase) Selection in Multistable Systems: Stochastic Resonance, Signal Bias, and the Effect of Signal Phase

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It is common for a periodically driven nonlinear system to have coexistent attractors. Specifically, we are interested in the case where one of these attractors is a limit cycle whose period is an integer multiple of that of the driving. Suppose that in a given application the system is desired to evolve with this limit cycle with a particular phase relative to the driving. In this Letter we propose a technique combining the effect of noise and the effect of a bias periodic signal with a properly chosen phase to accomplish this task.

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Introduction.—Periodically driven nonlinear oscillators constitute a class of models that underlies many important applications. It frequently occurs that, in the parameter region of interest, such a system exhibits several coexistent attractors (multistable). Let T denote the period of the forcing. Suppose, for the object of this Letter, that one of these attractors is periodic with mT ($m > 1$) its period. In a given situation this orbit may be wanted for its wave form and spectral properties. If the system is strobed at times $t_n = nT$ ($n = 0, 1, 2, 3, \ldots$) while moving on this attractor, we obtain m discrete surface of section points, repeating every m cycles of the forcing. Thus, starting at $t = 0$, from each of these m points we can have a distinct resulting trajectory, differentiated from one another by their relative behavior with respect to the driving, which we henceforth loosely refer to as the phase; each of these m trajectories has its own domain of initial conditions [1]. Clearly, the existence of multiple basins and domains can be detrimental in such applications as laser arrays or Josephson junctions, where a collection of nearly identical systems with weak coupling is often desired to evolve in synchrony. On the other hand, the richness provided by these phenomena, through proper means of utilization, can also lead to improved system performance.

The above problem was first noted by Pecora and Carroll, who in Ref. [2] proposed a method to drive a multistable system to a preselected trajectory, regardless of where the system is initialized. Their method involves adding a chaotic signal with certain periodic characteristics as part of the drive, and they showed that it works effectively when applied to electronic circuits. Our objective is to present a simpler and more flexible approach, combining the use of noise and a bias signal, to achieve the same result. Specifically, we show that while the random noise, through a mechanism similar to stochastic resonance, is important in eliminating unwanted basins, it is the small periodic bias signal that is crucial in attaining the desired domain. Furthermore, as an additional parameter, the phase of the bias can be tuned in such a way that any one of the m available relative trajectories is selected at will. For a spatially distributed ensemble of multistable systems, this aspect of the technique is especially useful in realizing not only coherent solutions of different phases but also patterns of requisite spatial modulations. We point out that in our approach the combination of noise and signal is activated only for a brief period of time, just long enough to drive an arbitrary initial condition to the target trajectory. After that, it is withdrawn and the system is left to evolve in its natural state. This is in contrast to the method of Ref. [2] where the chaotic drive remains on for as long as the system is operating (i.e., the system is permanently altered). We illustrate our ideas below using two examples, one a differential equation system and the other a circle map.

Example 1.—Consider the driven Duffing equation

$$\ddot{x} + 0.05\dot{x} + x^3 = a + b \cos t + [f_s(t) + f_n(t)], \quad (1)$$

where $f_s(t)$ is the bias signal and $f_n(t)$ represents the effect of random forcing. Specifically, we assume that $f_n(t)$ takes the form $f_n(t) = A_n \eta(t)$, where $\eta(t)$ is Gaussian white noise with $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\, \eta(s) \rangle = \delta(t - s)$. The signal $f_s(t)$ is chosen to be in the form $f_s(t) = A_s \sin(\omega t + \phi)$, where ω and ϕ are to be determined by the actual application.

For $f_s(t) = f_n(t) \equiv 0$, $a = 0.15$, and $b = 0.21$, Eq. (1) exhibits three periodic attractors, of period 1, 2, and 3, respectively. Consider a rectangle in the phase space $(x, \dot{x})$ defined by $-0.6 < x < 0.6$ and $-0.2 < \dot{x} < 0.2$.

We find that, starting at $t = 0$, about 20% of the points in this region are attracted to the period-3 attractor, while the remaining 80% converge to the period-2 attractor, with equal proportions in the two distinct trajectories denoted α and β. The overlap between the rectangle and the basin of attraction of the period-1 attractor is very small, accounting for less than 1% of the region, and we ignore it for the time being.

Now our task is to obtain a period-2 trajectory with a preselected phase (e.g., the phase α) irrespective of where the system is initialized. We use numerical experiments to illustrate our strategy for achieving this goal. To begin, consider a uniform grid of 70×70 initial conditions on the aforementioned rectangle. For each of the 4900 points, we evolve [3] Eq. (1) for $100T = 200\pi$ units of time, $T = 2\pi$ being the period of the driving, and then reduce the value of $f_s(t)$ and $f_n(t)$ linearly to zero for another $50T$. In the numerical work, this "quenching" process is implemented by setting A_s and A_n to ϵA_s and ϵA_n, and reducing ϵ from one to zero in a ramp. After the removal of $f_s(t)$ and $f_n(t)$, we examine the status of the resulting point $(x, \dot{x})$. The following terms are used for classifying the finding. The values of $L_2(0)$ and $L_3(0)$ denote the numbers of points in the basins of the period-2 and period-3 attractors at $t = 0$. The symbols L_2 and L_3 denote the same quantities at $t = 150T$. We define N_α and N_β to be the numbers of points in domain α and domain β of the period-2 attractor at $t = 150T$; clearly $N_\alpha + N_\beta = L_2$.

First, we study the role of noise. Let $f_s(t) = 0$. In Fig. 1, the horizontal axis is the strength of noise A_n and the vertical axis plots $\mu_1 = L_2/[L_2(0) + L_3(0)]$. As can be seen, the effect of noise increases with A_n, reaching an optimal level when $A_n \approx 0.005$, where almost 100% of the initial conditions are driven to the basin of the period-2 attractor. Upon further increase of A_n, however, the effectiveness of the random perturbation degrades monotonically. The tail portion of the curve indicates that, when the noise level becomes too large, more and more points are actually driven to the period-1 attractor. We note that the length of time for which the noise is activated can have important influence on the details of the curve in Fig. 1. Specifically, if noise is allowed to operate on a time scale much longer than that used in Fig. 1, a range of noise strength can be used to achieve the maximal result (see example 2 below).

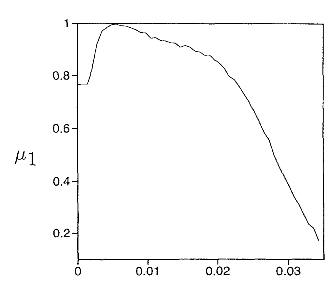

FIG. 1. Plot of $\mu_1 = L_2/[L_2(0) + L_3(0)]$ versus A_n for Eq. (1) with $A_s = 0$. Here μ_1 is the ratio of the number of points found in the basin of the period-2 attractor after the removal of perturbation to the total number of initial conditions in the rectangle $-0.6 < x < 0.6$ and $-0.2 < \dot{x} < 0.2$.

The above observation can be understood in simple terms by considering the one dimensional problem of viscous particles moving in a bounded potential with two minima denoted A and B. Say A is the target attractor which is more stable (i.e., A is a deeper minimum) than B. We draw an analogy between A and the period-2 attractor, and between B and the period-3 attractor. Within a given time scale, if the added random perturbation is weak, most particles initialized in B will remain in B, corresponding to the relatively small value of μ_1 seen in Fig. 1 for small A_n. As the noise becomes stronger, more and more particles will be driven to A, leading to the monotonic rise in the function of μ_1 versus A_n. This process continues until the optimal noise level is attained, beyond which the perturbed particles can jump more and more easily from A to B, giving rise to the monotonic decline portion of the μ_1 versus A_n curve in Fig. 1. If a third, even deeper minimum is present in the system, then large noise will ultimately drive the system toward this new state, corresponding to the tail portion of Fig. 1. Generally, the role of noise is twofold. It generates incoherent random jumps in all directions. But with proper strength, it also induces coherent motion from the less stable attractor (with "shallow" basin) to the more stable one (with "deep" basin). The competition between these two seemingly opposite effects yields the type of curve in Fig. 1. This mechanism is similar to that of stochastic resonance [4,5] where noise can be used to facilitate information processing [6].

Unlike basins of attraction, the definition of domains of initial conditions for periodically forced systems depends on the time when the observation is made. In the case of the period-2 attractor of the Duffing equation, a point in domain α at $t = 0$ maps to domain β at $t = T$, and vice versa. That is, the two domains are symmetric with respect to the time translation $t \rightarrow t + T$. This suggests that random, nondiscriminating perturbations will not be very useful by themselves in driving the system from one domain to another. Our numerical experiments confirm this idea. But imagine that if a judiciously chosen signal, such as an impulse train with period $2T$ and appropriate amplitude, is applied to the system, then the above symmetry is broken and it is conceivable that the trajectory may be led to the preferred domain after certain transient time. In fact, we find on our first choice that the most accessible signal, the sinusoid, works adequately for this purpose, provided its period is twice that of the driving. Specifically, we let $f_s(t) = A_s \sin(t/2 + \phi)$, where ϕ is a tunable phase parameter. Figure 2(a) plots

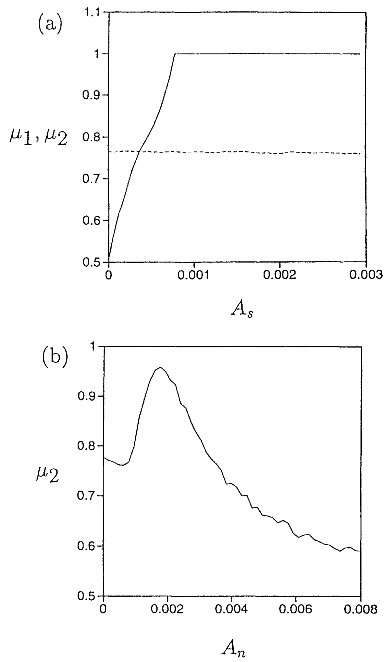

FIG. 2. (a) Plot of $\mu_2 = N_\alpha/[N_\alpha + N_\beta]$ versus A_s (solid line) and $\mu_1 = L_2/[L_2(0) + L_3(0)]$ versus A_s (dashed line) for Eq. (1) with $A_n = 0$ and $\phi = \pi/2$. (b) Plot of μ_2 versus A_n with $A_s = 0.0004$ and $\phi = \pi/2$. The quantity μ_2 gives the ratio of the number of points in domain α to the number of points in the basin of the period-2 attractor after the removal of perturbation.

$\mu_2 = N_\alpha/[N_\alpha + N_\beta]$ against A_s (solid line) with $f_n(t)$ set to zero and $\phi = \pi/2$. Evidently, a very small signal of $A_s = 0.0008$ is already sufficient to produce a 100% conversion from domain β to domain α. Under the same condition, the fraction $\mu_1 = L_2/[L_2(0) + L_3(0)]$ (dashed line) shows no significant change for the given range of A_s. This means that the added bias signal has almost no effect on the basins of the period-2 and period-3 attractors. Figure 2(b) shows the result of combining the signal and noise on the elimination of unwanted domains. We fix $A_s = 0.0004$ and $\phi = \pi/2$. The vertical axis is $\mu_2 = N_\alpha/[N_\alpha + N_\beta]$. Again, we observe a stochastic resonance type of response as A_n increases. Here the noise helps to achieve a higher percentage of conversion compared to the case with signal alone.

We now turn to the role played by the phase ϕ. The result is shown in Fig. 3, where we choose $A_s = 0.002$ and $A_n = 0.004$, and plot $\mu_3 = N_\alpha/[L_2(0) + L_3(0)]$ against ϕ. The main effect is that there are two disjoint intervals of ϕ values for which opposite effects are obtained in terms of selecting a specific domain. In particular, for $\pi/4 < \phi < 4\pi/5$, about 95% of the initial conditions in the rectangle are driven to domain α. In contrast, domain α captures almost no points if $11\pi/8 < \phi < 17\pi/10$. In other words, nearly all the trajectories, except a few that still linger on the period-3 attractor,

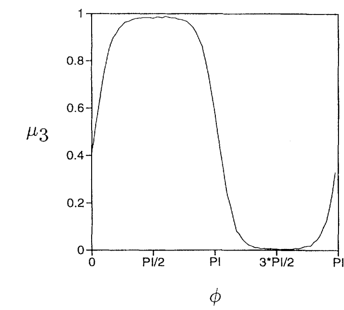

FIG. 3. Plot of $\mu_3 = N_\alpha/[L_2(0) + L_3(0)]$ versus ϕ for Eq. (1) with $A_s = 0.002$ and $A_n = 0.004$. Here μ_3 is the ratio of the number of points found in domain α of the period-2 attractor after the removal of perturbation and the total number of initial conditions in the rectangle $-0.6 < x < 0.6$ and $-0.2 < \dot{x} < 0.2$.

are moved to domain β. This flexibility is an important feature rendered by the phase of the bias signal.

Figure 4 illustrates the combined effect of noise and the bias signal. We let $\phi = 5\pi/6$ and plot $\mu_3 = N_\alpha/[L_2(0) + L_3(0)]$ versus A_n for different values of A_s. The four successively higher curves correspond to $A_s = 0.0004$, 0.001, 0.002, and 0.004. The optimal case occurs when $A_s = 0.004$ and $A_n \approx 0.0044$ where 99% of the points started in the rectangle $-0.6 < x < 0.6$ and $-0.2 < \dot{x} < 0.2$ are driven eventually to the period-2 trajectory of the phase α.

Example 2.—It is often the case that the qualitative dynamics of a periodically driven nonlinear oscillator like Eq. (1) can be adequately modeled by circle maps of the form

$$x_{m+1} = F(x_m) \mod 1\,, \tag{2}$$

where x is an angular variable and $F(x + 1) = F(x) + 1$. For the purpose of this paper we choose $F(x)$ to be $F(x) = x + A - Bg(x)$ with $g(x) = 4x$ for $0 \le x < 0.25$, $g(x) = 2 - 4x$ for $0.25 \le x < 0.75$, and $g(x) = 4x - 4$ for $0.75 \le x < 1$. The map with the addition of noise and a bias signal is

$$x_{m+1} = x_m + A - Bg(x_m) + A_n\eta_m + A_sS_m \mod 1\,. \tag{3}$$

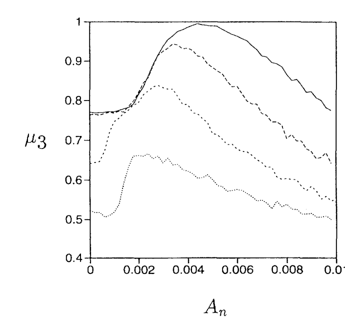

FIG. 4. Plot of $\mu_3 = N_\alpha/[L_2(0) + L_3(0)]$ versus A_n for Eq. (1) with $\phi = 5\pi/6$ where successively higher curves correspond to $A_s = 0.0004$, 0.001, 0.002, and 0.004.

As shown in Ref. [7], this map has a period-2 and a period-3 attractor coexisting for $A = 0.41$, $B = 0.29$, and $A_n = A_s = 0$. Our objective here is to drive an arbitrary initial condition to a particular phase of the period-2 orbit. To this end we assume η_m to be a random number uniformly distributed in $[-0.5, 0.5]$ and $S_m = 0$ if $m =$ odd and $S_m = -1$ if $m =$ even.

From Fig. 2(a) we observe that, if points are initialized in the basin of the period-2 orbit, then with signal alone we can achieve an 100% conversion to the desired phase. This, in combination with Fig. 1, suggests that the noise and the signal can also be applied in two consecutive stages: First, all the points are driven to the basin of the period-2 attractor with noise, and then these points are further moved to the appropriate domain with the signal. We demonstrate this approach with Eq. (3). In the numerical work we begin with 5000 initial conditions distributed uniformly in the unit interval. Then we evolve these points under Eq. (3) for 500 iterates. The next 50 iterates are used to remove the perturbation linearly. In Fig. 5 we show the result of $\mu_1 = L_2[L_2(0) + L_3(0)]$ versus A_n when $A_s = 0$. Note that, since in this case the time scale during which the noise is active is relatively long, a range of noise strength can be applied to achieve the maximal effect. Finally choosing $A_n = 0.015$ and $A_s = 0.005$ and activating the drive in two stages, we convert all the 5000 initial conditions to the desired phase of the period-2 orbit.

In summary, a technique is proposed which eliminates multiple basins as well as domains of attraction in periodically driven systems. The goal is to make the system behave with a periodic trajectory that has a preselected phase relative to the driving. There are three essential components in the technique: noise, a bias periodic signal, and the phase of the signal; the role of each is illustrated with numerical simulations performed on two examples. Our result demonstrates that the technique is flexible and could be easily implemented in experiments. We suggest that an important area of future application is spatiotemporal systems where nearly identical units with weak coupling are desired to evolve in synchrony [8].

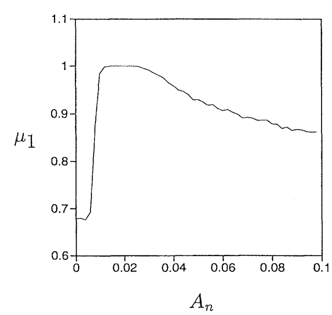

FIG. 5. Plot of $\mu_1 = L_2[L_2(0) + L_3(0)]$ versus A_n for Eq. (3) with $A_s = 0$.

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[1] We henceforth speak about basins and domains of attraction in terms of the surface of sections taken every period of the forcing starting at $t = 0$. Thus the basin of attraction of the period-m attractor is the union of its m domains.

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