Nonlinear Phenomena and Intermittency in Plasma Turbulence

B. Ph. van Milligen, C. Hidalgo, and E. Sánchez
Asociación EURATOM-CIEMAT, 28040 Madrid, Spain
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A new technique combining wavelet analysis and bispectral analysis has been developed. This analysis tool permits the detection of structure in turbulent or chaotic data with time resolution, even in the presence of a significant noise contribution. Application of this technique to data obtained in fusion plasmas with Langmuir probes demonstrates its possibilities by detecting short-lived intermittent nonlinear coupling. Its application in the field of chaos analysis is indicated.

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The purpose of this Letter is to introduce a new analysis tool for turbulent or chaotic data to the physics community. It allows detection and characterization of short-lived structures in turbulence. The characterization and understanding of strong turbulence is especially urgent in the field of thermonuclear plasma physics, where the so-called anomalous transport which deteriorates the energy confinement due to turbulence is an important phenomenon that is far from understood. Apart from the difficulty of obtaining local measurements of turbulent quantities in the hostile plasma environment, the main obstacle to analysis is the high fractal dimension of between 5 and 9 of the turbulence [1]. Most traditional methods for determining the nature of the turbulence rely to a large degree on correlation techniques, probability distributions, and spectral analyses, all involving long time averages [2]. Similar techniques supplemented with two-dimensional visualization have been reported in [3]. The wavelet bicoherence technique presented here detects phase coupling while reducing time averages to a minimum, thus permitting short-lived events, pulses, and intermittency to be resolved. Its use is especially indicated for signals contaminated with noise, although its application to data from numeric models without noise has also been highly successful.

Wavelet bicoherence is based on two existing techniques: wavelet analysis and bispectral analysis.

Wavelet analysis is a relatively recent technique [4–7] that has enjoyed increasing popularity in the field of chaos and turbulence. Rather than giving time-averaged estimates of the frequency contributions to a signal, as are provided by the traditional Fourier decomposition, the wavelet analysis decomposes a signal into wavelet components that depend on both scale (which under some conditions is equivalent to frequency) and time. The use of wavelet analysis is recommended in the analysis of data records containing pulses or short-lived events in order to avoid averaging out these temporally localized occurrences by examining large sections of the data record.

Whereas the Fourier decomposition is based on the harmonic wave $e^{i\omega t}$, the wavelet analysis is based on an *analyzing wavelet.* Many different wavelets are documented in the literature [4–8]. In the present work, we have chosen the following conceptually simple wavelet:

$$\Psi_a(t) = a^{-1/2} \exp[i2\pi t/a - (t/a)^2/2], \tag{1}$$

where a is the wavelet scale. This wavelet is a sinusoidal oscillation convoluted with a Gaussian having a half-width of $\Delta t = a$ (Fig. 1, top). The Fourier transform of this wavelet (Fig. 1, bottom) shows a single peak at a frequency $\omega = 2\pi/a$ with a FWHM value of $\Delta\omega = \omega/4$. Δt and $\Delta\omega$ are the time and frequency resolution of the wavelet and satisfy a Heisenberg relation, $\Delta t \Delta\omega = \pi/2$. The present choice of wavelet provides a reasonable compromise between time and frequency resolution, although other choices may be more suitable for different purposes. Because of the single peak in the frequency spectrum, a wavelet of scale a may be interpreted as representing a frequency $\omega = 2\pi/a \pm \Delta\omega/2$.

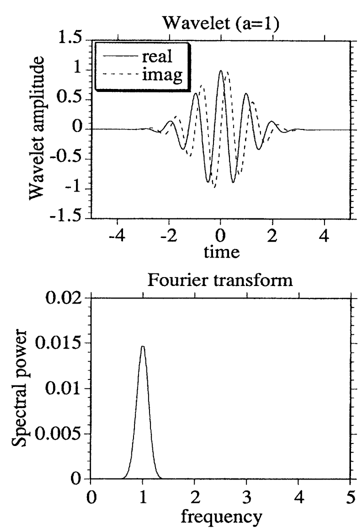

FIG. 1. Analyzing wavelet amplitude and Fourier transform at scale $a = 1$. Similar plots for wavelets at other scales can be obtained by means of the scaling transformations, $\Psi_a(t) = (1/\sqrt{a})\Psi(t/a)$ and $\hat{\Psi}_a(\omega) = \sqrt{a}\,\hat{\Psi}(\omega a)$, where $\hat{\Psi}$ is the Fourier transform of Ψ.

The wavelet transform of a function $f(t)$ is given by

$$W_f(a,\tau) = \int f(t)\Psi_a(t-\tau)\,dt\,, \qquad (2)$$

which is a function of both scale and time. More mathematical details of wavelet transforms can be found in the cited literature. We mention that, apart from the continuous wavelet transform as given by Eq. (2), the discrete or fast wavelet transform also exists for orthogonal wavelets [4,6,8]. The application of wavelets to bispectral analysis requires, however, the use of the continuous wavelet transform, and therefore we will not expand any further on this point.

Bicoherence.—The other ingredient of the new technique presented here is bispectral analysis [2,9,10]. The *bicoherence* is a measure of the amount of phase coupling that occurs in a signal. Phase coupling is defined to occur when two frequencies, ω_1 and ω_2, are simultaneously present in the signal(s) along with their sum (or difference) frequencies, and the sum of the phases ϕ of these frequency components remains constant. The bicoherence measures this quantity, being a function of two frequencies ω_1 and ω_2, which is close to 1 when the signal contains *three* frequencies ω_1, ω_2, and ω that satisfy the relation $\omega_1 + \omega_2 = \omega$ and $\phi_1 + \phi_2 = \phi +$ const; if no such relation is satisfied, it is close to 0. In other words, it measures the *coupling strength* between two frequencies and their sum (and difference) frequencies. When the turbulence exhibits *structure* of any kind whatsoever, it may be expected that some phase coupling occurs. It has been shown that the bicoherence is proportional to the *coupling constant* in some quadratic wave-interaction turbulence models (e.g., drift-wave turbulence in plasmas) [2,11].

Traditionally the bicoherence has been evaluated using the Fourier transform [2,9]. However, in many cases the application of Fourier analysis to turbulent data is unsatisfactory, because it presupposes the existence of *modes* in the physical system with well-defined frequencies ω that perdure in time. The chaotic systems under study are not expected to possess many modes, if any. In addition, the long time series that are necessary in the Fourier-based bicoherence analysis in order to have both sufficient frequency resolution and statistics [2] do not permit detection of intermittent behavior and average out many interesting turbulent effects.

We define the wavelet bispectrum as

$$B^w(a_1,a_2) = \int W_f^*(a,\tau)W_f(a_1,\tau)W_f(a_2,\tau)\,d\tau\,, \qquad (3)$$

where the integral is taken over a finite time interval T: $\tau_0 \leq \tau \leq \tau_1$, and

$$1/a = 1/a_1 + 1/a_2 \qquad \text{(frequency sum rule)}. \qquad (4)$$

The wavelet bispectrum measures the amount of phase coupling in the interval T that occurs between wavelet components of scale lengths a_1, a_2, and a of $f(t)$ such that the sum rule is satisfied. Since the scale lengths may be interpreted as inverse frequencies, $\omega = 2\pi/a$, one may interpret the wavelet bispectrum as the coupling between wavelets of frequencies such that $\omega = \omega_1 + \omega_2$, within the frequency resolution.

The squared wavelet bicoherence is the normalized squared bispectrum,

$$[b^w(a_1,a_2)]^2 = \frac{|B^w(a_1,a_2)|^2}{[\int |W_f(a_1,\tau)W_f(a_2,\tau)|^2\,d\tau][\int |W_f(a,\tau)|^2\,d\tau]}\,, \qquad (5)$$

which can attain values between 0 and 1.

The squared bicoherence $[b^w(a_1,a_2)]^2$ is usually plotted in the (ω_1,ω_2) plane rather than the (a_1,a_2) plane for ease of interpretation. There is no need to represent the whole plane due to the symmetries in the definition and the limitation set by the Nyquist frequency [9].

It is convenient to introduce the summed bicoherence, which is defined as $[b^w(a)]^2 = \sum[b^w(a_1,a_2)]^2$ where the sum is taken over all a_1 and a_2 such that Eq. (4) is satisfied, and the total bicoherence $(b^w)^2 = \sum[b^w(a)]^2$, where the sum is taken over all a. Naturally, the numerical values of these quantities depend on the chosen calculation grid and are therefore not fundamental. But they serve to compare cases computed under the same numerical conditions since they summarize the information conveniently, as will be seen later.

Error estimation.—For digitally sampled signals, the integrals in Eqs. (3) and (5) are replaced by summations over N points. Each factor that is an integral over T then suffers an error of $1/\sqrt{N}$, so that the error in the bicoherence is estimated by

$$\Delta[b^w(\omega_1,\omega_2)]/b^w(\omega_1,\omega_2) \approx 2/\sqrt{N}\,. \qquad (6)$$

Noise level.—The wavelet coefficients are not statistically independent, since we have chosen to use wavelets that are not orthogonal. Each wavelet coefficient is calculated by evaluating Eq. (2), integrating over the range $-\infty < t < \infty$. Because of the periodicity a of the wavelets of scale a (cf. Fig. 1), two statistically independent estimates of the wavelet coefficients are separated by a time $a/2$ or by a number of points $M(a) = a\omega_{\mathrm{samp}}/4\pi$ (ω_{samp} being the sampling frequency). Thus, the summation done in the evaluation of the bicoherence $b^w(\omega_1,\omega_2)$ is not carried out over N independent estimates, but only over $N[\max M(a)]$, where the maximum is taken over the values of a that come into play for the evaluation of a specific value of the squared bicoherence. An upper bound for the statistical noise level in $b^w(\omega_1,\omega_2)$ is, therefore,

$$\varepsilon[b^w(\omega_1,\omega_2)] \approx \left[\frac{\omega_{\mathrm{samp}}/2}{\min(|\omega_1|,|\omega_2|,|\omega_1+\omega_2|)}\,\frac{1}{N}\right]^{1/2}. \qquad (7)$$

This is an upper bound because the statement that two independent wavelet coefficient estimates are a time $a/2$ apart is a maximum estimate. Observe that at low frequencies the statistical noise may dominate the bicoherence, and a significant interpretation must limit itself to (relatively) high frequencies. At higher frequencies, however, the noise level drops rapidly with N, and in this sense the wavelet bicoherence may be considered a very powerful noise filter for coherent signals.

Intermittency.—We have analyzed data taken with Langmuir probes in the edge region of the Advanced Toroidal Facility (ATF). This is an $l = 2, M = 12$ torsatron with major radius $R_0 = 2.10$ m and minor radius $a = 0.27$ m. The observed edge plasma turbulence has been characterized using the techniques described elsewhere [12]. The plasma is heated by electron cyclotron resonant heating with a total power of $P_{\mathrm{ECH}} \approx 200$ kW, has an average electron density of $\overline{n}_e \approx (4 - 6) \times 10^{12}$ cm^{-3}, a stored energy $S_E \approx 1$ kJ, and a magnetic field $B = 1$ T. Figure 2 shows results for the measured floating potential (related to the electron temperature T_e) at $Z/Z_{\mathrm{shear}} \approx 0.85$ (Z_{shear} being the location of the velocity shear layer). The signal, sampled at 1 MHz with 10-bit resolution, was analyzed on the time window from 1 to 15 ms by subdividing the signal into 28 sections of 0.5 ms each. For each section (of 500 points) the wavelet bicoherence was calculated on a frequency grid with a separation between grid points of 10 kHz. The top graph of Fig. 2(a) shows the time development of the total bicoherence. The noise level according to Eq. (7) is indicated by the dashed line, and the error according to Eq. (6) is indicated by the error bars. The short bursts of coherency in this graph are an indication of intermittency in the phase coupling processes occurring in the turbulent plasma. Four sections of 0.5 ms have been selected to illustrate this time development in more detail. The lower part of Fig. 2(a) shows the summed bicoherence for these sections, where again the dashed line is the noise level and the error bars are indicated. Figure 2(b) shows the bicoherence on the same four sections. Observe the large amount of detail: The strong bicoherence at low frequencies associated with

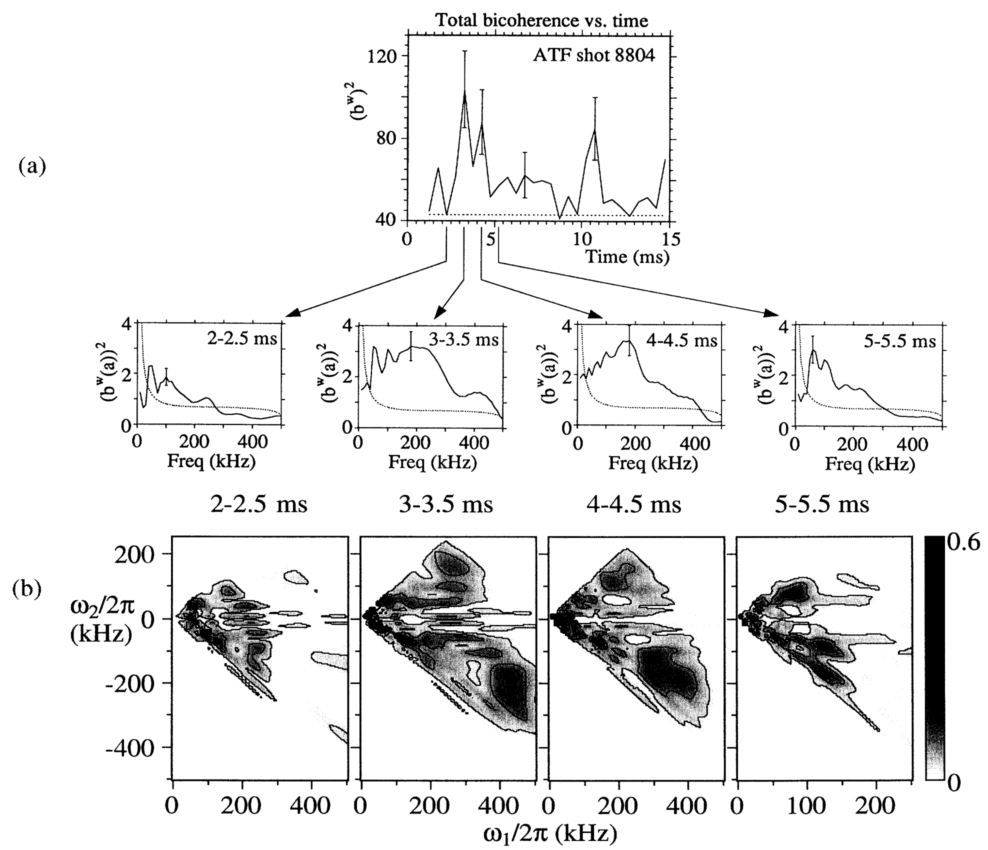

FIG. 2. (a) Analysis of Langmuir probe data of an ATF discharge with strong overall Fourier bicoherence. The total wavelet bicoherence shown in the upper graph is strong on average and exhibits a pronounced time dependence. The noise level is indicated by the dashed line and the error bars are shown. This is an example of turbulent intermittency. The summed bicoherence graphs at four selected time intervals of 0.5 ms each indicate that the maxima in the bicoherence are mainly due to sum frequencies in the range 150–250 kHz. (b) Bicoherence plots at four time intervals as in (a). Note the strong coupling at high frequencies in the second and third graphs, responsible for the intermittent behavior, and the horizontal and diagonal ridges present in all graphs but especially apparent in the plot of 3–3.5 ms, indicative of coupling of a localized mode to broad-spectrum turbulence.

horizontal and diagonal ridges is due to the coupling of single modes to broadband spectra (this is especially clear in the section 3–3.5 ms) which occurs throughout the sequence, while the main contribution to the intermittent behavior comes from the diffuse structures at high frequencies.

Detection of structure in turbulence.—The possibility of detecting structure in turbulence depends, among other things, on the time resolution. We have analyzed Langmuir probe data of an ATF discharge for which the bicoherence calculated with the Fourier-based method [2] exhibits a relatively low overall value. Figure 3(a) shows results for the measured floating potential at $Z/Z_{\mathrm{shear}} \approx 0.9$. A data record running from 1 to 16 ms has been analyzed. Figure 3(a) shows the summed bicoherence graphs for both a long (1 to 16 ms) and a short (10 to 10.5 ms) time window and their corresponding noise levels. The long time average produces a very low bicoherence level (although still significantly above noise level), because the turbulent structures that are present are not constant in time. The short time average, however, shows a rather strong bicoherence, indicating the detection of a turbulent structure around $t = 10$ ms. The two peaks at 60 and 120 kHz indicate a structure of a particular size. Figure 3(b) shows the two-dimensional bicoherence graphs for both time intervals. Indeed, a clear structure is visible in the short interval.

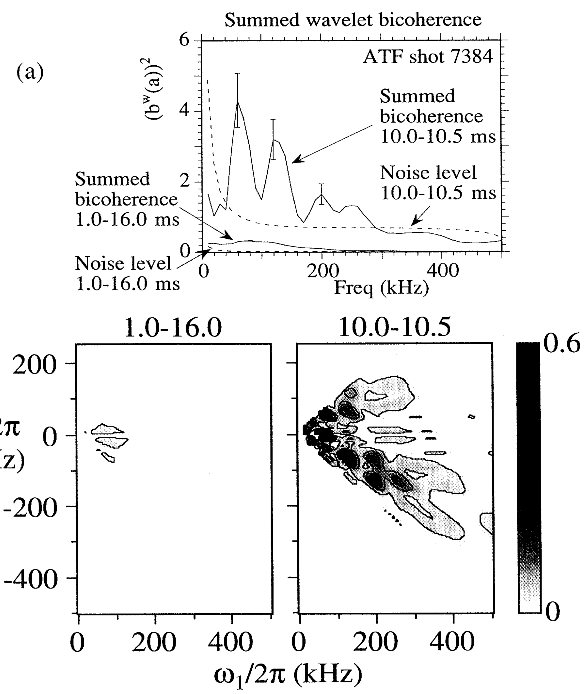

FIG. 3. (a) Summed bicoherence for two time windows, 1.0 to 16.0 and 10.0 to 10.5 ms. The noise level for both cases is indicated by the dashed lines. The two peaks at 60 and 120 kHz in the short time section indicate the detection of structure in the turbulence. (b) Bicoherence graphs for the time windows 1.0 to 16.0 and 10.0 to 10.5 ms. A structure is clearly visible in the latter.

Conclusions.—In this Letter we have introduced a new technique for the analysis of turbulent data. The traditional bicoherence based on Fourier analysis detects phase coupling between oscillations. The new wavelet bicoherence adds time resolution to this technique without sacrificing the interpretation in terms of frequencies by selecting a suitable analyzing wavelet. This permits the resolution of structure in turbulent or chaotic data that would otherwise go unnoticed. In particular, the detection of intermittency and structure has been demonstrated in turbulent plasmas. The characterization of turbulence or chaos by means of the wavelet bicoherence may provide a link between theory and experiment that allows the exploration of the dynamics of chaos with higher fractal dimensions than has been possible hitherto.

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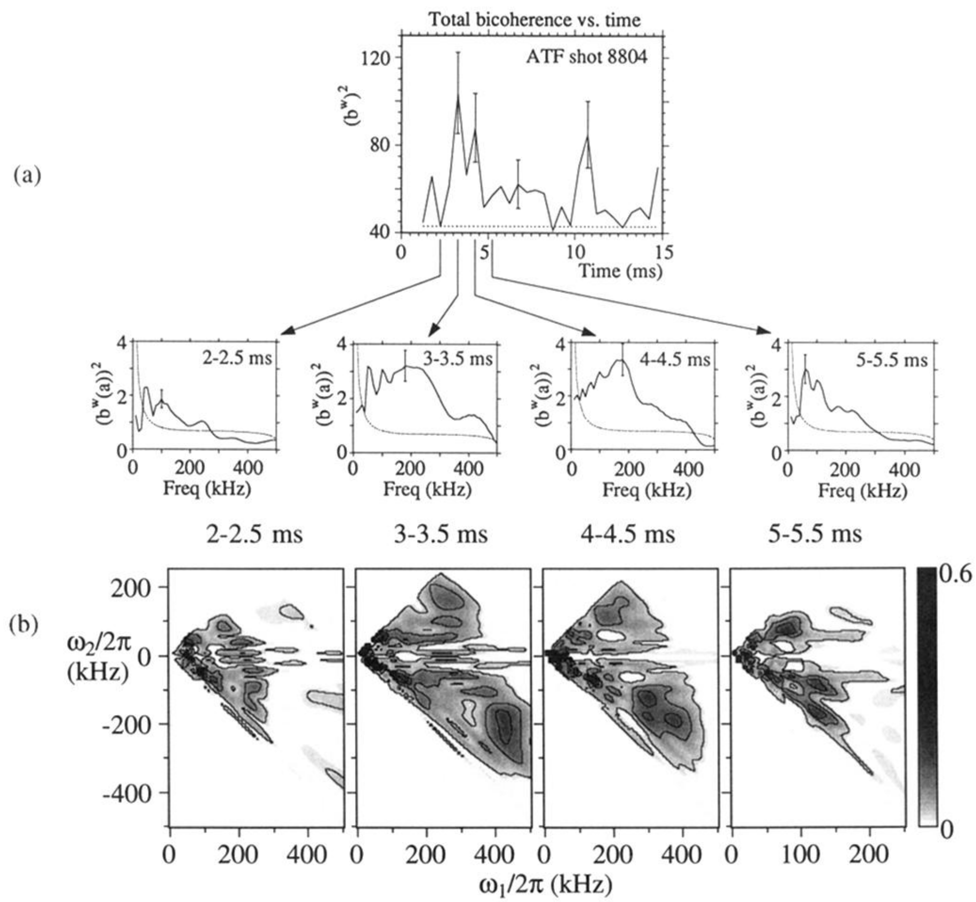

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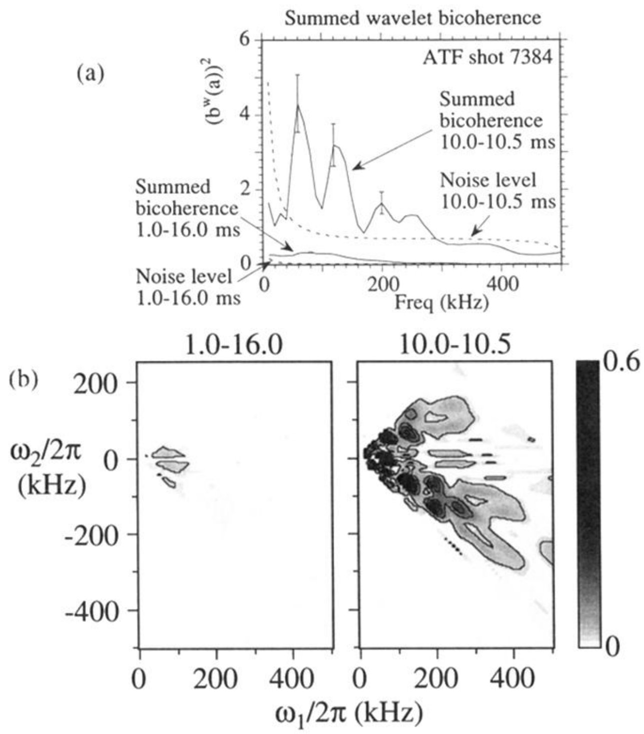

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