Vacuum Structure and Spectrum of $N = 2$ Supersymmetric SU(n) Gauge Theory

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We present an exact description of the metric on the moduli space of vacua and the spectrum of massive states for four dimensional $N = 2$ supersymmetric SU(n) gauge theories. The moduli space of quantum vacua is identified with the moduli space of a special set of genus $n - 1$ hyperelliptic Riemann surfaces.

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Recently Seiberg and Witten [1] obtained exact expressions for the metric on moduli space and dyon spectrum of $N = 2$ supersymmetric SU(2) gauge theory using a version of Olive-Montonen duality [2]. In this Letter we use this approach to obtain similar information for the $N = 2$ supersymmetric SU(n) gauge theory.

The $N = 2$ Yang-Mills theory involves a single chiral $N = 2$ superfield in the adjoint which decomposes into an $N = 1$ vector multiplet W_α and chiral multiplet Φ. W_α includes the gauge field strength $F_{\mu\nu}$ as well as the Weyl gaugino, while Φ includes a Weyl fermion and a complex scalar ϕ. Classically there is an ($n - 1$)-complex-dimensional moduli space of flat directions. Any vacuum expectation value (VEV) for ϕ can be rotated by a gauge transformation to lie in the Cartan subalgebra of SU(n), and generically breaks SU(n) $\rightarrow$ U(1)$^{n-1}$. Denote by Φ_i and W_i the components of the chiral and vector superfields in the Cartan subalgebra with respect to the same basis. The low-energy effective action $S \sim \mathrm{Im}(\int d^2\theta\, d^2\overline{\theta}\, \Phi_D^i \overline{\Phi_i} + \frac{1}{2}\int d^2\theta\, \tau^{ij} W_i W_j)$ is derived from a single holomorphic function $\mathcal{F}(\Phi_k)$ since $\Phi_D^i = \partial\mathcal{F}/\partial\Phi_i$, and $\tau^{ij} = \partial^2\mathcal{F}/\partial\Phi_i\,\partial\Phi_j$ [3]. The real and imaginary parts of the lowest component of τ^{ij} are the effective θ angles and coupling constants of the theory, respectively.

Normalize the W_i fields so that the charges of fields in the fundamental of SU(n) form a unit cubic lattice, implying that the allowed set of electric charges n_e^i are all the integers. Denoting the magnetic charges of any monopoles or dyons by $2\pi n_{m,i}$, the Dirac quantization condition requires the $n_{m,i}$ to lie in the dual lattice, implying that the $n_{m,i}$ are also integers. Then the effective action is left invariant by duality transformations [1,4] which acts on the fields by $\mathbf{\Phi} \rightarrow \mathbf{M} \cdot \mathbf{\Phi}$, $\mathbf{W} \rightarrow \mathbf{M} \cdot \mathbf{W}$ for $\mathbf{M} \in \mathrm{Sp}(2n - 2, \mathbb{Z})$, where ${}^t\mathbf{\Phi} = (\Phi_D^i, \Phi_i)$, ${}^t\mathbf{W} = (W_D^i, W_i)$, and W_D^i are the dual U(1) field strengths. The structure of the $N = 2$ supersymmetry algebra implies [5] dyons of magnetic and electric charge ${}^t\mathbf{n} = (n_{m,i}, n_e^i)$ have mass $M = \sqrt{2}({}^t\mathbf{a} \cdot \mathbf{n})$, where ${}^t\mathbf{a} = (a_D^i, a_i)$ are the VEVs of the scalar component of the chiral superfield and its dual: $a_i = \langle\phi_i\rangle$ and $a_D^i = \langle\phi_D^i\rangle$.

As discussed in Ref. [1], analyticity of the superpotential $\mathcal{F}$, positivity of the Kähler metric Imτ, and the form of the superpotential at weak coupling imply that there must be singularities in the moduli space around which the theory has nontrivial monodromies in Sp($2n - 2, \mathbb{Z}$).

Classical moduli space.—We adopt the convention that upper case indices $I, J, K, \ldots$ run from 1 to n and lower case indices $i, j, k, \ldots$ run from 1 to $n - 1$. Use a basis $\{H^I, E_\pm^{IJ}\ (I > J)\}$ for the generators of the U(n) Lie algebra where the $n \times n$ matrices $[H^I]_{AB} = \delta_A^I \delta_B^I$ span the Cartan subalgebra. Then the SU(n) vector superfield $W = W_I H^I + W_{IJ}^\mp E_\pm^{IJ}$ will satisfy the tracelessness condition $\sum_I W_I = 0$. We everywhere substitute for W_n in terms of the W_i's using the tracelessness constraint, thus choosing the W_i's as a basis of the Cartan subalgebra of SU(n) and ensuring that the charges of fields in the fundamental of SU(n) are integers.

Rotate $\langle\phi\rangle$ into the Cartan subalgebra of SU(n): $\langle\phi\rangle = a_I H^I$, where the a_I satisfy the tracelessness constraint $\sum_I a_I = 0$. If we denote the space of independent complex a_I's by $\mathcal{T}_n$, then the classical moduli space is $\mathcal{M}_n = \mathcal{T}_n/S_n$ where S_n is the Weyl group of SU(n) which acts by permuting the a_I's. The Higgs mechanism gives the $W_{IJ}^\pm$ boson masses proportional to $|a_I - a_J|$. The Weyl group S_n does not act freely on $\mathcal{T}_n$: a submanifold of partial symmetry breaking to SU(m) is fixed by $S_m \subset S_n$, since m of the a_I's are equal there. Classically $\mathcal{M}_n$ has singularities along these submanifolds since extra $W_{IJ}^\pm$ bosons become massless there. A global U(1)$_\mathcal{R}$ symmetry of the SU(n) theory is broken down to $\mathbb{Z}_{4n}$ by anomalies. Since the scalar field ϕ has charge 2 under this symmetry, only a $\mathbb{Z}_{2n}$ acts nontrivially on $\mathcal{M}_n$.

A basis of gauge-invariant coordinates covering $\mathcal{M}_n$ at weak coupling are given by $u_\alpha = \langle\mathrm{Tr}(\phi^\alpha)\rangle = \sum_I a_I^\alpha$, for $\alpha = 2, \ldots, n$. A more convenient set of gauge-invariant coordinates is given classically by the elementary symmetric polynomials in the a_I's,

$$s_\alpha \equiv (-)^\alpha \sum_{I_1 < \cdots < I_\alpha} a_{I_1} \cdots a_{I_\alpha}, \quad \alpha = 1, \ldots, n. \qquad (1)$$

These symmetric coordinates can be expressed as polynomials in terms of the u_α's generated by Newton's formula $r s_r + \sum_{\alpha=0}^{r} s_{r-\alpha} u_\alpha = 0$, for $r = 1, 2, 3, \ldots$, where $s_0 \equiv 1$, $u_0 \equiv 0$, and $s_1 = u_1 = 0$ by the tracelessness constraint.

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The SU(n) curve.—The couplings $\boldsymbol{\tau}$ transform under $\mathrm{Sp}(2n-2,\mathbb{Z})$ and $\mathrm{Im}\boldsymbol{\tau}$ must be positive definite for the theory to be unitary. The period matrix of a genus $n-1$ Riemann surface has precisely these properties, so it is natural to guess [1] that the moduli space of the $\mathrm{SU}(n)$ theory be embedded in the moduli space of the Riemann surface. A simple set of Riemann surfaces are the hyperelliptic ones [6], described by the complex curve $y^2 = \prod_{\ell=1}^{2n}(x - e_\ell)$, which is the double-sheeted cover of the Riemann sphere branched at $2n$ points e_ℓ. The $\mathrm{SU}(n)$ curve should also have a $\mathbb{Z}_{2n}$ symmetry, reflecting the $\mathrm{U}(1)_{\mathcal{R}}$ symmetry broken by instantons in the $\mathrm{SU}(n)$ theory. This symmetry fits naturally with the hyperelliptic surfaces if we assign $\mathcal{R}$ charge 1 to x and n to y.

We now assume, following [7], that the coefficients of the polynomial in x defining the $\mathrm{SU}(n)$ curve are themselves polynomials in the gauge-invariant coordinates s_α (or u_α) and Λ_n^{2n}, where Λ_n is the renormalization scale of the $\mathrm{SU}(n)$ theory. The power of Λ_n^{2n} ensures that it has the quantum numbers of a one-instanton amplitude.

In the weak coupling limit there are nontrivial monodromies around the regions of moduli space where extra gauge symmetries are restored, corresponding to the submanifolds where a pair or more of the a_I take the same values. So, as $\Lambda_n \to 0$, the $\mathrm{SU}(n)$ curve should be singular along these submanifolds. A curve is singular whenever a pair or more of its branch points e_ℓ coincide. A polynomial in x which has the required property is $F(x) = \prod_{I=1}^{n}(x - a_I)$. As we will shortly see, there is also a monodromy of the $\mathrm{SU}(n)$ theory at weak coupling which does not correspond to any coincidence of the a_I's. Thus, in the weak coupling limit the $\mathrm{SU}(n)$ curve should be singular for *all* values of the a_I's. This can be achieved by simply squaring the polynomial $F(x)$, so that all its zeros are doubled. Also, it then has the right degree in x to desribe a hyperelliptic curve. There is then only one way to add in instanton contributions (terms dependent on Λ_n) consistent with our assignment of the $\mathcal{R}$ charges: $y^2 = F^2(x) - \Lambda_n^{2n}$. The coefficients of the polynomial $F(x)$ are precisely the elementary symmetric functions s_α of the a_I's (1). We make the assumption that the s_α remain good global coordinates on the $\mathrm{SU}(n)$ moduli space even at strong coupling. Then the proposed $\mathrm{SU}(n)$ curve is

$$y^2 = \left(\sum_{\alpha=0}^{n} s_\alpha x^{n-\alpha}\right)^2 - \Lambda_n^{2n}. \qquad (2)$$

The remainder of this Letter describes various consistency checks of this curve. For brevity's sake, we check only properties that depend on the conjugacy class of the monodromies in $\mathrm{Sp}(2n-2,\mathbb{Z})$.

Weak coupling monodromies.—The first check we perform is to show that (2) has all the right monodromies at weak coupling. $\mathrm{SU}(n)$ is strongly broken down to $\mathrm{SU}(n-1)$, when $a_i \sim a$ and $a_n \sim (1-n)a$, where $|a| \gg \Lambda_n$, and the usual renormalization group matching implies $\Lambda_n^{2n} \sim a^2 \Lambda_{n-1}^{2(n-1)}$. Shifting x to $x + a$ and taking the limit $a \to \infty$ while leaving Λ_{n-1} fixed sends two branch points to infinity. Rescaling y by $(x + na)^{-1}$, we recover the curve (2) again, but now for $\mathrm{SU}(n-1)$ instead of $\mathrm{SU}(n)$. Thus the $\mathrm{SU}(n)$ curve at weak coupling automatically contains all $\mathrm{SU}(n-1)$ monodromies, allowing us to proceed by induction in n.

The SU(2) curve can be shown to be equivalent to the SU(2) curve found in [7] by a fractional linear transformation on the x variable, since the automorphisms of the Riemann sphere allow us to fix three of the branch points arbitrarily by an $\mathrm{SL}(2,\mathbb{C})$ transformation. Along an SU(2) direction at weak coupling the SU(3) curve degenerates to the SU(2) curve, and so gives the correct monodromies. The SU(3) curve has another singularity at weak coupling corresponding to the limit where all the a_I's scale together by some large factor. If the special SU(3) monodromy around this singularity agrees with the answer calculated from perturbation theory, then all the weak coupling monodromies of SU(3) will have been checked, and the induction can proceed to SU(4), etc. In general, we will need to compute just one special monodromy for each $\mathrm{SU}(n)$ curve.

A convenient curve along which to measure this monodromy is

$$a_I(t) = \omega^{I+t} a, \quad 0 \le t \le 1, \qquad (3)$$

where $|a| \gg \Lambda_n$ and $\omega = e^{2\pi i/n}$. This path traverses a large circle in the s_n complex plane along which all the other s_α's vanish. In this plane the $\mathrm{SU}(n)$ curve (2) factorizes for $|s_n| \gg \Lambda_n^n$ as $y^2 = \prod_J (x - \omega^J s_n^{1/n}[1 + s_n^{-1}\Lambda_n^n])(x - \omega^J s_n^{1/n}[1 - s_n^{-1}\Lambda_n^n])$. The branch points are arranged in n pairs with a pair at each nth root of unity times $s_n^{1/n}$. As $s_n \to e^{2\pi i} s_n$, these pairs are rotated into one another in a counter-clockwise sense, and each pair also revolves once about its common center in a clockwise sense.

Choose cuts and a standard basis for the independent cycles on the $\mathrm{SU}(n)$ surface as shown for SU(3) in Fig. 1. Thus, γ_i are independent nonintersecting cycles, similarly for γ_D^i, and their intersection form is $(\gamma_D^i, \gamma_j) = \delta_j^i$. Note

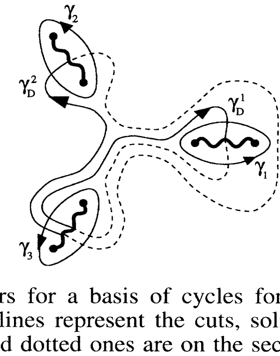

FIG. 1. Contours for a basis of cycles for the SU(3) curve. The thick wavy lines represent the cuts, solid contours are on the first sheet, and dotted ones are on the second.

that γ_n is not independent of the γ_i's: a simple contour deformation shows that $\sum_I \gamma_I = 0$.

As $s_n \to e^{2\pi i} s_n$ the γ_I are dragged along with the cuts so that $\gamma_i \to \gamma_{i+1} \equiv P_i^j \gamma_j$, where $P_i^j = \delta_{i+1}^j - \delta_{i+1}^n$ is a representation of the $\pi = (1, \ldots, n)$ permutation. It then follows from the defining properties of symplectic matrices that the monodromy $\boldsymbol{\gamma} \to \mathbf{M} \cdot \boldsymbol{\gamma}$ in $\mathrm{Sp}(2n - 2, \mathbb{Z})$ of ${}^t\gamma = (\gamma_D^i, \gamma_i)$ can be written as

$$\mathbf{M} = \begin{pmatrix} \mathbf{1} & \mathbf{N} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} {}^t\mathbf{P}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{pmatrix}, \tag{4}$$

where $\mathbf{P}$ is the permutation matrix found above, and $\mathbf{N}$ is some symmetric matrix which we wish to determine. If $\mathbf{NP} = {}^t\mathbf{P}^{-1}\mathbf{N}$, the two matrices in Eq. (4) commute, and $\mathbf{M}^n = \left(\begin{smallmatrix} \mathbf{1}_n & \mathbf{N} \\ \mathbf{0} & \mathbf{1} \end{smallmatrix}\right)$ since $\mathbf{P}^n = \mathbf{1}$. But $\mathbf{M}^n$ is easy to compute: as $s_n \to e^{2\pi i n} s_n$, the γ_i cycles are simply dragged back to themselves and similarly for the γ_D^i cycles except that their ends get wound n times (in a clockwise sense) around each cut that they pass through. As illustrated in Fig. 2, each such winding can be deformed to give two of the associated γ_i's. Keeping track of the signs, one finds $\gamma_D^i \to \gamma_D^i - 2n(\gamma_i - \gamma_n) \equiv \gamma_D^i + nN^{ij}\gamma_j$, where $N^{ij} = -2(\delta^{ij} + 1)$. Since this matrix satisfies $\mathbf{NP} = {}^t\mathbf{P}^{-1}\mathbf{N}$, it follows that it is, in fact, the matrix $\mathbf{N}$ of Eq. (4).

Special monodromies in perturbation theory.—By asymptotic freedom, $\mathrm{SU}(n)$ is weakly coupled if it is broken at a large scale so that all the $|a_I - a_J| \gg \Lambda_n$. Writing the effective action as $S \sim \int \tau^{IJ} W_I W_J$, the one-loop result for the running of the couplings is $\tilde{\tau}^{IJ} = (i/\pi)(\delta^{IJ} \sum_K \ln a_{IK} - \ln a_{IJ})$, where $a_{IJ} \equiv a_I - a_J$. The tracelessness constraint $\sum_I W_I = 0$ implies $\tau^{ij} = \tilde{\tau}^{ij} - \tilde{\tau}^{in} - \tilde{\tau}^{nj} + \tilde{\tau}^{nn}$. It follows from the definition of τ^{ij} that $a_D^i = \tau^{ij} a_j$ (only in perturbation theory).

In general, there is a nontrivial monodromy in the a_D^i along any path in $\mathcal{M}_n$ at weak coupling whose lift to $\mathcal{T}_n$ connects a point with its image under the action of a nontrivial permutation $\pi \in S_n$. The different possible choices of permutation π reflect the pattern of symmetry breaking of $\mathrm{SU}(n)$ at high energies, except for $\pi = (1, \ldots, n)$, which does not correspond to any special symmetry breaking pattern. The associated monodromy is the one special to $\mathrm{SU}(n)$.

Along the path (3) realizing this monodromy, the a_i's transform as $a_i \to P_i^j a_j$, where $\mathbf{P}$ is the same permutation found above from the curve. The logarithms in the one loop expression for τ^{ij} contribute a shift in its monodromy, $\tau^{ij} \to \tau^{ij} + N^{ij}$, where $\mathbf{N}$ is easily computed to be equal to the matrix $\mathbf{N}$ found above from the curve. The a_D^i then transform as $a_D^i \to \tau^{ij} P_j^k a_k + N^{ij} P_j^k a_k$. From the defining properties of symplectic matrices it follows

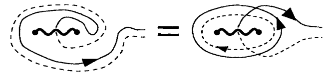

FIG. 2. A contour wound once around a cut is deformed into the sum of three contours.

that $\boldsymbol{\tau P} = {}^t\mathbf{P}^{-1}\boldsymbol{\tau}$, and so the monodromy of the scalar VEVs ${}^t\mathbf{a} = (a_D^i, a_i)$ indeed agrees with the monodromy (4) computed from the $\mathrm{SU}(n)$ curve. This completes our check that the monodromies of the curve (2) agree with all the monodromies of the $\mathrm{SU}(n)$ theory at weak coupling.

Metric on moduli space and dyon spectrum.—The identification of the metric and spectrum—that is to say, a_i and a_D^i as functions of the moduli s_α—closely parallels the discussion of Ref. [1]. Choosing a basis of cycles (γ_D^i, γ_i) of the $\mathrm{SU}(n)$ curve with the canonical intersection form $(\gamma_D^i, \gamma_j) = \delta_j^i$, we identify a_i and a_D^i as sections of a flat $\mathrm{Sp}(2n - 2, \mathbb{Z})$ bundle over moduli space given by

$$a_i = \oint_{\gamma_i} \lambda, \quad a_D^i = \oint_{\gamma_D^i} \lambda, \tag{5}$$

where λ is some meromorphic one-form on the curve with no residues. There is a $(2n - 2)$-dimensional space of such forms spanned by the $n - 1$ holomorphic one-forms $\lambda_i = (x^{i-1}/y)\,dx$, and the $n - 1$ meromorphic one-forms $x^n \lambda_i$. The one-form λ defining our solution can be written as a linear combination of these basis one-forms (with coefficients that can depend on the s_α and Λ_n) up to a possible total derivative.

Since the period matrix of the Riemann surface defined by the $\mathrm{SU}(n)$ curve has a positive definite imaginary part, transforms in the same way as τ^{ij} under $\mathrm{Sp}(2n - 2, \mathbb{Z})$, and has the same monodromies as τ^{ij} does, it follows that they should be identified. Since the period matrix and the couplings are defined by $\tau^{ij}(\oint_{\gamma_j} \lambda_\alpha) = \oint_{\gamma_D^i} \lambda_\alpha$ and $\tau^{ij}(\partial a_j/\partial s_\alpha) = (\partial a_D^i/\partial s_\alpha)$, respectively, it follows that $\partial a_i/\partial s_\alpha = \oint_{\gamma_i} \lambda_\alpha$ and $\partial a_D^i/\partial s_\alpha = \oint_{\gamma_D^i} \lambda_\alpha$ for some basis λ_α of holomorphic one-forms. Equation (5) and these identifications imply a set of differential equations for λ, which can be solved to find [8]

$$\lambda \propto \left(\sum_{\alpha=0}^{n} (n - \alpha) s_\alpha x^{n-\alpha} \right) \frac{dx}{y}, \tag{6}$$

since $\partial\lambda/\partial s_\alpha = -x^{n-\alpha}(dx)/y + d(x^{n+1-\alpha}/y)$. The overall constant normalization of λ can be determined only by making a choice of basis cycles and matching to perturbation theory.

Strong coupling monodromies.—The singularities of the curve (2) occur where a pair or more of the branch points coincide, and correspond to a dyon in the spectrum becoming massless. Near these points in moduli space the low-energy U(1) that couples to the massless dyon flows to zero coupling. Thus, there will be a dual description of the physics near the singularities which is weakly coupled, and so can be used to check these limits of the curve (2) as well.

Consider the case where m dyons become massless at a point P in $\mathcal{M}_n$. The low-energy theory is by definition local, so all m massless dyons must be mutually local, implying their charge vectors $\mathbf{n}^a$ are symplectically orthogonal: ${}^t\mathbf{n}^a \cdot \mathbf{I} \cdot \mathbf{n}^b = 0$ for all $a, b = 1, \ldots, m$, where $\mathbf{I}$ is the symplectic form $\left(\begin{smallmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{smallmatrix}\right)$. This can only be satisfied for

$m \leq n - 1$ linearly independent vectors since there exists a symplectic transformation to dual fields where each dyon is described as an electron charged with respect to only one dual low-energy U(1). In this dual description m independent electrons are becoming massless near P. These m electrons are massless along intersecting hypersurfaces given by the vanishing of m dual scalar VEVs: $\tilde{a}_a = 0$.

Along these hypersurfaces the effective action is singular, leading to nontrivial monodromies for paths looping around them. The one-loop effective couplings near P are $\tilde{\tau}^{ij} = (-i/2\pi)\delta^{ij}(\tilde{n}_e^i)^2 \ln(\tilde{n}_e^i a_i)$, where $\tilde{n}_e^i$ denotes the charge of the ith electron, implying the monodromy $\mathbf{M}_i$ around the $\tilde{a}_i = 0$ hypersurface to be

$$\mathbf{M}_i = \begin{pmatrix} \mathbf{1} & (\tilde{n}_e^i)^2 \mathbf{e}_{ii} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \tag{7}$$

where $\mathbf{e}_{ii}$ is a matrix of zeros except for a 1 in the ith position along the diagonal. A strong coupling test of the curve (2) is that its monodromies around intersecting singular submanifolds all be conjugate to the above $\mathbf{M}_i$ monodromies corresponding to mutually local dyons.

The singular submanifolds of the SU(3) curve are given by the vanishing of the discriminant Δ of the polynomial (2) defining the curve. Possible intersection points of the singular submanifold $\Delta = 0$ are at *its* singular points. There are five such points: the $\mathbb{Z}_3$-symmetric triplet of points $4s_2^3 = -27\Lambda^6$ and $s_3 = 0$, and the $\mathbb{Z}_2$ doublet $s_2 = 0$ and $s_3^2 = \Lambda^6$. The triplet corresponds to true intersection points. At the $\mathbb{Z}_2$ points, however, $\Delta = 0$ describes a branch point of a single submanifold, instead of the intersection point of two submanifolds.

We compute the monodromies around the intersecting singular submanifolds at a $\mathbb{Z}_3$ point by first expanding the SU(3) curve in local coordinates around one such point, where it is found that two pairs of branch points coincide. Choose a basis of γ_i cycles to encircle each pair of branch points, and the γ_D^i's in the canonical way. The resulting monodromies computed by dragging and deforming contours are then found to be of the form (7) with $\tilde{n}_e^1 = \tilde{n}_e^2 = 1$. This confirms that there are indeed two different mutually local dyons becoming massless along the two intersecting submanifolds at the $\mathbb{Z}_3$ points. Furthermore, their charges are consistent with the semiclassically stable dyon charges in the SU(2) limit. This suggests that, as in the SU(2) case, the spectrum of stable dyon charges remains the semiclassical one all the way down to these strong-coupling singularities.

As a final check of the SU(3) curve, we note that the $\mathbb{Z}_3$ intersection points imply the known $N = 1$ SU(3) vacuum structure. Following Ref. [1], add to the microscopic $N = 2$ theory a mass term for the $N = 1$ chiral superfield Φ which breaks $N = 2$ to $N = 1$. Going to the dual (weakly coupled) description of the physics near a point in the moduli space of the SU(n) theory where $n - 1$ dyons are massless, and minimizing the nonperturbative [9] effective superpotential shows that the $N = 2$ flat directions are lifted and only the point where all $n - 1$ dyons are massless remains an $N = 1$ vacuum. The three $\mathbb{Z}_3$ singularity intersection points of the SU(3) curve found above are just such points, and happily they correspond to the three $N = 1$ SU(3) vacua related by a spontaneously broken $\mathbb{Z}_3$. Finding the strong coupling singularities for the SU(n) curve becomes increasingly difficult for higher n.

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