Two-Dimensional String-Theory Model with No Folds

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We propose a string theory model which explains several features of two-dimensional Yang-Mills theory. Folds are suppressed. This in turn leads to the empty theory in flat target spaces. The Nambu-Goto action appears in the usual way. The model naturally splits into two (chiral) sectors: orientation preserving maps and orientation reversing maps. Moreover, it has a straightforward extension to three- and four-dimensional space-times.

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It is strongly believed that the dynamics of gauge fields can be described in terms of a string theory. The idea was supported by the lattice strong coupling expansion [1] and the $1/N_c$ expansion [2]. The latter applied in two-dimensional (2D) models gave several well-established relations between QCD in two dimensions (or Yang-Mills theory in two dimensions) and a string theory [3,4]. It appeared that the crucial role is played by the no-fold condition, which strongly restricts the set of the surface-to-surface maps defining the string theory. Moreover, the results of [3,4] indicate that the proper string action should contain the Nambu-Goto term. It is well known that the Nambu-Goto term alone cannot give the correct picture because the appropriate functional integral cannot suppress folds.

In this Letter we propose a solution to this problem. We shall supplement the Nambu-Goto action by a topological term which will lead to the cancellation of folds. The topological term is well defined in a target space-time of dimension four so we shall introduce two additional (hereafter called vertical) variables with values in R^2. They will enter only the topological term—in this sense they will not have any dynamics. Functional integration over vertical variables will lead to the nonperturbative cancellation of folds. We shall show that the model has null partition functions and null transition amplitudes for microscopic states (infinitesimal punctures) for strings propagating in the flat 2D target space-time. At the end of this paper we shall comment on implications of these results for the physical, 4D string model of gauge fields.

The topological term we are going to consider is the self-intersection number (I) [5] of a surface immersed in the extended 4D space-time. It was previously considered in [6–9]. It is a topological invariant, in some respects, similar to $F\tilde{F}$ of Yang-Mills theory in four dimensions: For example, the string analog of the $U_A(1)$ anomaly is proportional to I [7,8].

We depart from the Polyakov picture of string theory [10] in the sense that no dependence on the elementary world-sheet metric is involved—the only world-sheet metric we use is the induced one. It is worth noting that the area-preserving diffeomorphism plays no role in the whole construction.

The string theory functional integral for 2D targets is a sum over surface-to-surface maps $\Sigma \to M$, where Σ denotes the string world sheet and M the 2D target space-time. It is known that generic surface-to-surface maps contain singularities which are folds and cusps [11]. Hereafter we are going to consider only such maps. Folds form a submanifold of Σ for which one eigenvalue of the induced metric $g_{ab} = \partial_a \vec{X}\, \partial_b \vec{X}$ vanishes. We introduce two additional, (vertical) world-sheet fields: $(X^3, X^4) \in R^2$ and consider all lifts of the (X^1, X^2) map, i.e., maps of the form $(X^1, X^2, X^3, X^4) \in M \times R^2$, where fields (X^3, X^4) take arbitrary values. Generically, a lift is an immersion. It means that the image of Σ has two linearly independent tangent vectors so, e.g., the induced metric is nondegenerate. Folds are places where the surface-to-surface map ceases to be an immersion.

The proposed string action is

$$S[X] = \mu \int_M d^2\sigma \sqrt{g} + i\theta I[X], \tag{1}$$

where I is the self-intersection number of the surface immersed in the 4D space. For topologically trivial target spaces it equals

$$I = -\frac{1}{16\pi} \int_M d^2\xi \sqrt{g} g^{ab}\, \partial_a t_{\mu\nu}\, \partial_b \tilde{t}^{\mu\nu}, \tag{2}$$

where $\vec{X} = \vec{X}(\xi)$ defines the immersion and $t_{\mu\nu} = \epsilon^{ab}\, \partial_a X^\mu \partial_b X^\nu / \sqrt{g}$. The vertical coordinates enter the action only through I.

The action (1) is invariant under arbitrary v-regular homotopy of vertical fields: $X^\mu(\xi) = \epsilon^\mu(\xi)$ ($\mu = 3, 4$) [5]. A v-regular homotopy is a homotopy (i.e., just mentioned shift symmetry) which is an immersion for each homotopy parameter.

In the following, we are going to classify topological sectors of the model. We say that two immersions are in the same topological sector if they can be connected by a v-regular homotopy. Consider a 2D surface with folds and its lift into 4D space $(X^1, \ldots, X^4)$. The claim is that to any fold [in (X^1, X^2) space] corresponds infinitely many topologically inequivalent lifts characterized by a

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set of integer numbers. The latter are assigned to folds and are invariant under v-regular homotopies. A sum of these numbers gives the self-intersection number $I \in Z$. We stress again that all lifts have the same dependence on X^1, X^2, e.g., the same area [the same value of the Nambu-Goto part of the action (1)].

Consider a map of a sphere S^2 on R^2 with one fold S_1 (see Fig. 1). Thus the map is 2 to 1 everywhere except at the fold S_1 itself. The fold is topologically a circle S^1. One can always choose one nonvanishing, continuous tangent vector along the fold. It is the nonzero eigenvalue of the induced metric. Lifts of the fold must have nondegenerate 2D tangent space. Hence another vector tangent to the immersed surface has to be in the (X^3, X^4) plane. All the possible lifts belong to homotopy classes of the maps $S^1 \rightarrow S^1$, i.e., $\pi_1(S^1) = Z$. The first S^1 is the fold; the second S^1 represents nonzero tangents in the (X^3, X^4) plane. If σ parametrize the fold then for given lift $(X^3(\sigma), X^4(\sigma))$ the element of $\pi_1(S^1)$ is given by the rotation number of this tangent: $f = (1/2\pi) \int d\sigma(\partial_\sigma X^3\, \partial_\sigma^2 X^4 - \partial_\sigma X^4\, \partial_\sigma^2 X^3)/[(\partial_\sigma X^3)^2 + (\partial_\sigma X^4)^2]$. The integer f is invariant under the v-regular homotopy and is directly related to the self-intersection number I of the lifted configuration. We can see it if we notice that both numbers are additive under gluing. By gluing we mean a procedure of cutting small disks in both immersions and then connecting them by a tube. Let us associate a pair (f, I) of numbers to a lift. If we glue it with the (f', I') lift, we obtain the $(f + f', I + I')$ lift. Thus gluing f_1 copies of the $(-1, -I_1)$ lift with the $(f_1, 1)$ lift we get the $(0, 1 - f_1 I_1)$ lift. But $f = 0$ corresponds to $I = 0$, because the above simple map with $f = 0$ can be lifted to an immersion in just 3D space instead of 4D space. Thus $1 = f_1 I_1$, so $I_1 = f_1 = \pm 1$ (the sign is undetermined). The same reasoning can be repeated for more complicated folds with several disconnected components and cusps. One assigns the numbers $f_i \in \pi_1(S^1)$ to the ith connected component of the set of folds. The numbers $\{f_i\}$ are invariant under the v–regular homotopy. The self-intersection number is then $I[f] = \Sigma_{\text{folds}} \pm f_i$. One can see it gluing lifts of the just considered map with one fold to lifts of the other folds.

Below we give a simple proof of the following statement. *v-regular homotopy classes of lifts (i.e., topological sectors of the model) are in one-to-one correspondence with sets $\{f_i\}$.*

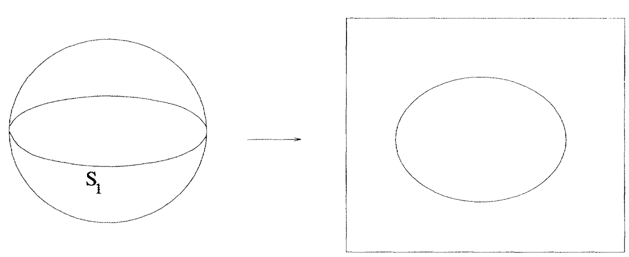

FIG. 1. A map of S^2 on R^2 with one fold S_1.

Let us consider two lifts (X^1, X^2, X_1^3, X_1^4) and (X^1, X^2, X_2^3, X_2^4). If they are v-regularly homotopic then they define the same set $\{f_i\}$, because $\{f_i\}$ is invariant under any v-regular homotopy. On the other hand, let us assume that two immersions are characterized by the same set $\{f_i\}$. Let S_1 denote folds of the map (X^1, X^2). The assumption implies that tangents to both lifts at S_1 are equal up to a v-regular homotopy—here it is a local rotation of the tangents. Moreover, one can shift the lifts in such a way that they will be equal $[(X_1^3, X_1^4) = (X_2^3, X_2^4)]$ at S_1. Hence both lifts are v-regularly homotopic in an infinitesimally small neighborhood of S_1 (up to second power of an infinitesimal quantity). Away from the folds the map (X^1, X^2) is an immersion. In this case the shift parameters $\epsilon^\mu(\xi)$ ($\mu = 3, 4$) can be arbitrary. Thus both lifts are v-regularly homotopic everywhere.

Now we go to the string theory. We want to show that the originally folded configurations (X^1, X^2) will cancel out from the partition function. Fixing the shift symmetry in such a way that the gauge slice picks only one representant (or at least an equal number of them) of each topological sector we get the following expression for the functional integral:

$$\int \mathcal{D}X^1\, \mathcal{D}X^2\, e^{-S[X]} \sum_{\{f_i\}} e^{i\theta I[f]}. \tag{3}$$

The sum over f_i's can be performed independently for each i because $I[f] = \Sigma_{\text{folds}} \pm f_i$. For one fold we get

$$\sum_{f \in Z} e^{\pm i\theta f} = 2\pi\delta(\theta). \tag{4}$$

Thus all folded configurations vanish from the path integral for nonzero θ. Of course, $\theta = \pi$ is the preferred value because then the model does not break parity.

Maps contributing to the vacuum-to-vacuum amplitude of the closed string necessarily have folds for the target space R^2. According to the above discussion the amplitude vanishes. This also holds for any correlation function of any finite set of local operators. Thus the final conclusion of this part of the paper is that the model (1) is trivial for the R^2 space-time.

In the end let us discuss shortly the Nambu-Goto action for maps without folds which may occur for topologically nontrivial space-times [in this case one can distinguish two classes of maps: orientation preserving (+) and orientation reversing maps (−)]:

$$\int_M d^2\sigma \sqrt{g} = \pm \frac{1}{2} \int_M d^2\sigma\, \epsilon^{\mu\nu}\, \epsilon^{ab}\, \partial_a X^\mu\, \partial_b X^\nu, \qquad \mu, \nu = 1, 2. \tag{5}$$

The sign is chosen in such a way that the right hand side is positive. In this way the Nambu-Goto action has been split into two (chiral) sectors of [4]. To some extent both sectors can be considered separately. In fact, this kind of theory has been considered recently in [12] yielding very interesting results.

Because the vertical degrees of freedom have no dynamics, one can view (1) as a compactification of a 4D string. In usual compactification schemes the vertical variables assume values in a compact 2D surface (e.g., torus). We believe that the cancellation of folds holds also in this case although we have not proved it. We must also stress that we can say very little about nongeneric surface-to-surface maps. Various subsets of these maps may be relevant for the description of various 2D gauge theories [13,14]. This will be considered in a separate publication. It is clear that the model (1) has straightforward extensions to three- and four-dimensional space-times; one simply needs to make the additional dimensions dynamical, i.e., add them to the Nambu-Goto action. Higher-dimensional string models may require more terms (e.g., the extrinsic curvature term [9]) which are ill defined or just do not exist for 2D space-times.

Concluding, we want to stress that arguments given in this paper show that the self-intersection number should play a major role in any 4D string theory of gauge fields. It is highly plausible that it will cancel all degenerate surface configurations which contribute to other string theories, e.g., the Polyakov string [10].

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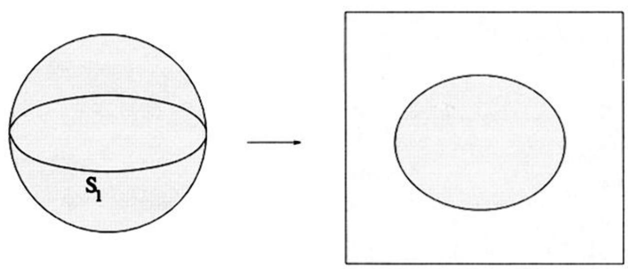

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