Universal Fluctuations in Spectra of the Lattice Dirac Operator

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Recently, Kalkreuter obtained the complete Dirac spectrum for an SU(2) lattice gauge theory. We performed a statistical analysis of his data and found that the eigenvalue correlations can be described by the Gaussian symplectic ensemble. Long range fluctuations are strongly suppressed: The variance of a sequence of levels containing n eigenvalues on average is given by $(1/2\pi^2)(\ln n + \text{const})$. Our findings are in agreement with the antiunitary symmetry of the lattice Dirac operator for $N_c = 2$ with staggered fermions. For $N_c = 3$ we predict that the eigenvalue correlations are given by the Gaussian unitary ensemble.

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The QCD Dirac operator is of fundamental importance for the calculation of the physical properties of QCD. Knowledge of its eigenvalues and eigenfunctions determines the propagator, a necessary ingredient for the calculation of hadronic correlation functions. In this Letter we will focus on the eigenvalues of the Dirac operator and isolate some universal properties that can be understood from its symmetries only. One such example is already known: The eigenvalues near zero satisfy sum rules [1] with a generating function given by random matrix theories with global symmetries of QCD [2]. This led us to the conjecture that the fluctuations of the eigenvalues no more than a few level spacings away from zero, over the ensemble of gauge field configurations, are given by universal functions that can be obtained from a much simpler random matrix theory. This raises the question whether the eigenvalues in the bulk of the spectrum show such universal characteristics as well.

Recently, in a ground breaking work, Kalkreuter [3] succeeded to compute the complete spectrum of the Dirac operator on a reasonably large lattice (12^4). His results were in complete agreement with an analytical sum rule adding greatly to our confidence in the accuracy of his results. Long level sequences have been analyzed before in atomic and nuclear physics and for systems with only a few degrees of freedom [4,5]. Generically, it was found that if the system is classically chaotic, the correlations between eigenvalues with the same exact quantum numbers are given by the Gaussian random matrix ensembles in spite of the fact that their average eigenvalue density is different.

In this Letter we will perform a statistical analysis of the lattice spectra using such methods [5]. We start from the assumption that the average eigenvalue density, $\overline{\rho}(\lambda)$, can be separated from the fluctuations of the eigenvalues about their average position. This allows us to unfold the spectrum. This is a procedure in which the eigenvalues are rescaled according to the average local level density. Formally, the unfolded spectrum $\{\lambda_n'\}$, with average eigenvalue density $\overline{\rho}'(\lambda) = 1$, is given by

$$\int_{-\infty}^{\lambda_n} \overline{\rho}(\lambda)\, d\lambda = \lambda_n' . \tag{1}$$

From the analysis of the spectra of the Hamiltonians of classically chaotic systems, we have learned that depending on the time reversal symmetry of the system the level correlations fall into three different universality classes. We want to stress that an *antiunitary* symmetry determines whether the matrix elements are complex, real, or quaternion real [6]. The corresponding invariant random matrix ensembles are called the Gaussian unitary ensemble (GUE), the Gaussian orthogonal ensemble (GOE), and the Gaussian symplectic ensemble (GSE), respectively. According to general universality arguments [7] the correlations between eigenvalues in the *bulk* of the spectrum, as opposed to those near the edge of the spectrum, are not sensitive to many other details of the random matrix model. For example, eigenvalue fluctuations in the bulk of the spectrum of random matrix models [2], with the chiral symmetry of the Dirac operator built in, will be given by one of the invariant random matrix models (GUE, GOE, or GSE).

Let us analyze the antiunitary symmetries of the Euclidean Dirac operator, $\mathcal{D} \equiv i\gamma_\mu \partial_\mu + \gamma_\mu A_\mu$, for fundamental fermions in an arbitrary SU(N_c) background gauge field A_μ. For three or more colors there are no antiunitary symmetries, and the matrix elements of the Dirac operator $\mathcal{D}$ are complex. For two colors the situation is different. In the continuum theory we have [1]

$$[\mathcal{D}^{\text{cont}}, C\tau_2 K] = 0 , \tag{2}$$

where C is the charge conjugation matrix ($C = \gamma_2\gamma_4$), τ_2 is one of the Pauli spin matrices, and K is the charge conjugation operator. Naive lattice fermions also obey this symmetry, but for Wilson fermions it is violated by the r term. For staggered lattice fermions the only remnant of the γ matrices is a phase factor ± 1 and instead of (2) we have [8]

$$[\mathcal{D}^{\text{stag}}, \tau_2 K] = 0 . \tag{3}$$

The important difference is that

$$(C\tau_2 K)^2 = 1, \quad \text{but } (\tau_2 K)^2 = -1 . \tag{4}$$

From a similar analysis of the time reversal operator in quantum mechanics (see [9]) we conclude that in the

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continuum theory the matrix elements of the Dirac operator can be chosen *real*, whereas for staggered fermions they can be organized into *real quaternions*. Therefore, we expect that the level correlations of the eigenvalues of $\mathcal{D}^{\text{cont}}$ are described by the GOE, whereas for $\mathcal{D}^{\text{stag}}$ they are given by the GSE. A necessary condition in both cases is that the gauge potential is "sufficiently random." For Wilson fermions the matrix elements are complex even for two colors, but at present it is not clear whether breaking the antiunitary symmetry by an irrelevant operator leads to different level statistics.

In his work, Kalkreuter [3] gives results for only a few SU(2) gauge field configurations, namely for $\beta = 4/g^2 = 1.8$ and $\beta = 2.8$, both with periodic and antiperiodic boundary conditions in the Euclidean time direction (in the spatial directions he uses periodic boundary conditions). His configurations have been obtained for dynamical staggered fermions with a mass m of $2ma = 0.2$ (a is the lattice spacing). For $\beta = 1.8$ the theory is certainly in the strong coupling phase, but it is generally believed that asymptotic scaling has been reached for $\beta = 2.8$.

With the eigenvalues of only one configuration we are unable to perform an ensemble average of their correlations. Instead we will perform a spectral average, which, at least in random matrix theory can be shown to be equal to the ensemble average [10]. For the present data we have verified numerically that the eigenvalue correlations do not change over the range of the spectrum.

The integrated level density (see Fig. 1)

$$N(\lambda) = \int_0^{\lambda} \rho(\lambda')\, d\lambda' \tag{5}$$

of the eigenvalues of $\mathcal{D}^{\text{stag}}$ follows immediately from the 10 368 eigenvalues calculated by Kalkreuter [3]. Except for a possible nonanalytical behavior at the ends of the spectrum, it is extremely smooth and almost linear. To unfold the spectrum we cut it in pieces of 500 eigenvalues and fit them by a second order polynomial. We have checked that this results in an average eigenvalue density equal to 1, and that our results are insensitive to the details of the cuts.

To define our statistics we introduce the quantity

$$N(x, x+n) = \int_x^{x+n} \rho'(\lambda)\, d\lambda\,, \tag{6}$$

where $\rho'(\lambda)$ is the unfolded spectral density. So, $N(x, x+n)$ is the number of eigenvalues in a sequence of length n starting at x. The *number variance* is defined by

$$\Sigma_2(n) = \frac{1}{p}\sum_{i=1}^{p}[N(x_i, x_i+n) - n]^2, \tag{7}$$

where points x_i are regularly spaced such that the sequences $[x_i, x_i + n]$ cover the complete spectrum. It can be related to the two-point level correlation function which is known analytically in random matrix theory (see [11]). For a random sequence of levels (Poisson spectrum) it can be shown that $\Sigma_2(n) = n$ (see [5]).

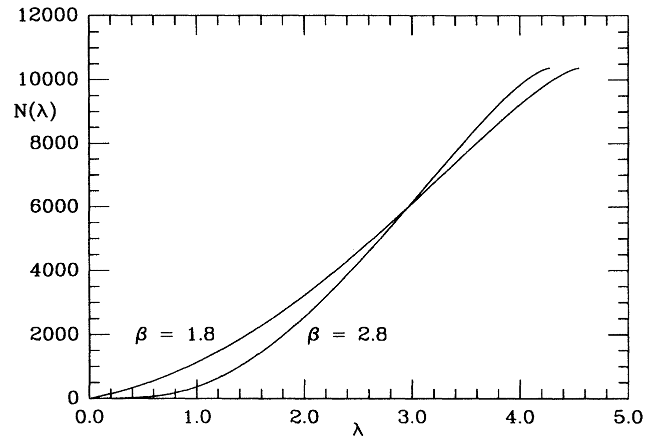

FIG. 1. The integrated eigenvalue density $N(\lambda)$ of the Euclidean lattice Dirac operator for SU(2) color with staggered fermions. Each curve is the result for one equilibrated gauge field configuration with β as indicated in the figure.

Our results for $\Sigma_2(n)$ are shown in the upper parts of Fig. 2 ($\beta = 1.8$) and Fig. 3 ($\beta = 2.8$). (To eliminate inaccuracies in the unfolding procedure we excluded 500 eigenvalues at the ends of the spectrum.) Long range fluctuations are almost completely absent. Instead of a variance of 100 for an average sequence of length 100 we find a variance of only 0.4, showing the presence of very strong correlations between the eigenvalues. From what we have said before, we expect that they can be described by the GSE. Indeed, the theoretical result (full curve) for $r \geq 1$ given by

$$\Sigma_2(r) = \frac{1}{2\pi^2}\left(\ln(4\pi r) + \gamma + 1 + \frac{\pi^2}{8}\right) + O\left(\frac{1}{\pi^2 r}\right), \tag{8}$$

shows perfect agreement with the lattice data. For comparison, we have also given the result for the GUE (dashed curve). The result for the GOE is for the most part outside the range of the figure.

A much smoother statistic is the Δ_3 statistic originally introduced by Dyson and Mehta [12]. It is related to the number variance by [13]

$$\Delta_3(r) = \frac{2}{r^4}\int_0^r (r^3 - 2r^2 s + s^3)\,\Sigma_2(s)\, ds\,, \tag{9}$$

and does not receive contributions from the quadratic term in $\Sigma_2(s)$ so that small unfolding errors are eliminated. Asymptotically, for $r \geq 1$, one finds for the GSE

$$\Delta_3(r) = \frac{1}{2}\Sigma_2(r) - \frac{9}{16\pi^2}\,. \tag{10}$$

Results for Δ_3 are shown in the middle parts of Figs. 2 and 3. Numerical results obtained from the data of

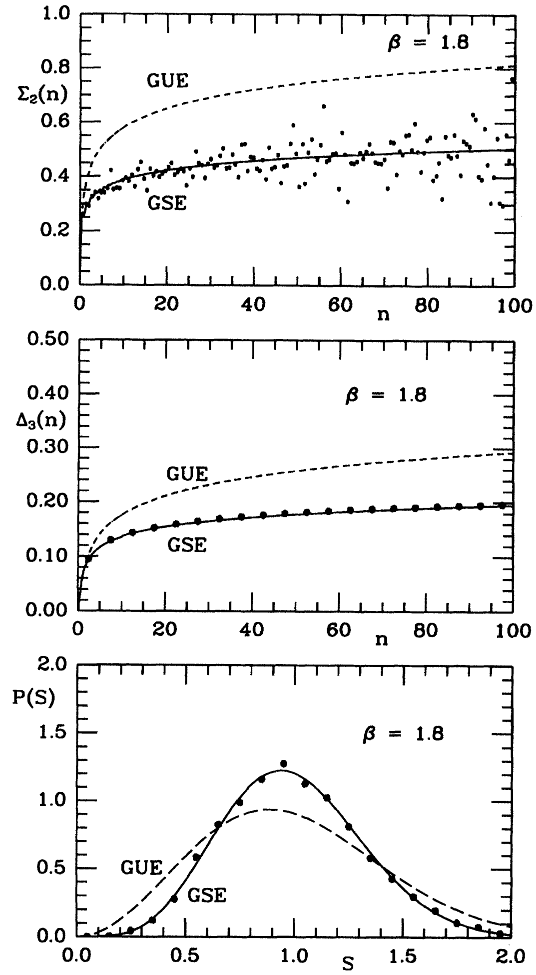

FIG. 2. Results for the level statistics $\Sigma_2(n)$ (upper), $\Delta_3(n)$ (middle), and the nearest neighbor spacing distribution $P(S)$ (lower) of the eigenvalues shown in Fig. 1 for $\beta = 1.8$. The random matrix results for the GSE and the GUE are represented by the full and the dashed line, respectively.

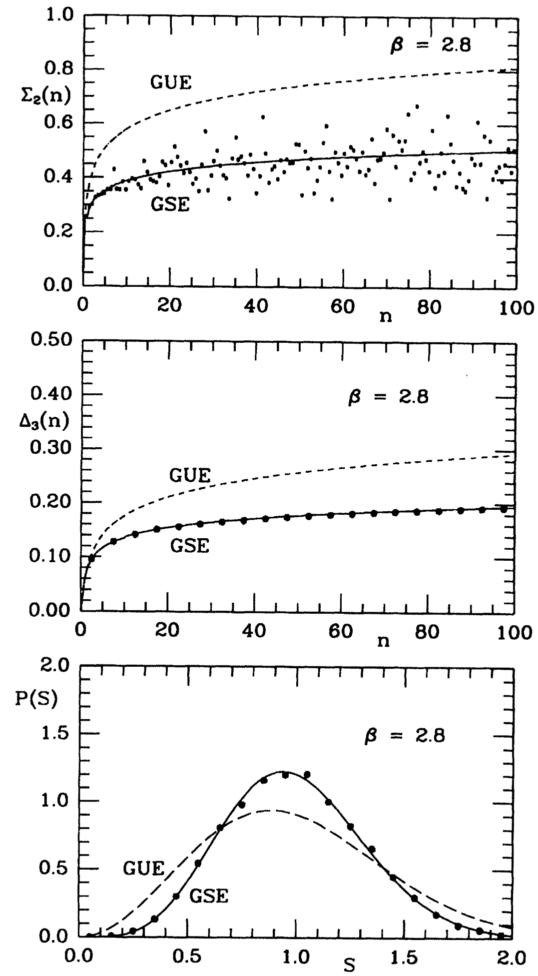

FIG. 3. Results for the level statistics $\Sigma_2(n)$ (upper), $\Delta_3(n)$ (middle), and the nearest neighbor spacing distribution $P(S)$ (lower) for $\beta = 2.8$. For further explanation see the caption of Fig. 2.

Kalkreuter [3] are given by full dots, whereas the full curve refers to the GSE result and the dotted curve represents the Δ_3 statistic of the GUE. Also in this case the curve for the GOE lies well above the GUE.

Finally, in the lower parts of Figs. 2 and 3, we show the distribution $P(S)$ of the nearest neighbor spacings $S = |\lambda'_{i+1} - \lambda'_i|$ of the unfolded spectrum. The histograms obtained from the lattice data are represented by points. The full lines show the GSE result for the nearest neighbor spacing distribution, and the dashed curve gives the result for the GUE. The exact expression is very well approximated by the Wigner surmise which for the GSE is given by

$$P(S) = \frac{2^{18}}{3^6\pi^3} S^4 \exp\left(-\frac{64}{9\pi} S^2\right). \tag{11}$$

For comparison we also show the result for the GUE (dashed curve) which is quadratic in S for small S. Concerning the spacing distribution of the GOE we only remark that it starts out linearly and deviates even more from the GSE. The nearest neighbor spacing distribution contains information on short range correlations in the spectrum. We hope that we have convinced the reader that they are also given by the GSE.

We have also analyzed the spectra that Kalkreuter obtained for periodic boundary conditions in all directions and found identical results for the eigenvalue correlations.

In conclusion, we have found that the fluctuations of the eigenvalues of the staggered Euclidean Dirac operator for SU(2) color can be described by the Gaussian symplectic ensemble both for $\beta = 1.8$ and $\beta = 2.8$. For the continuum theory the antiunitary symmetry is different, and we expect fluctuations according to the Gaussian orthogonal ensemble. In [2,14] we have shown that antiunitary symmetries are essential in determining the Goldstone sector of the theory. Because the massless sector for staggered fermions differs from the continuum theory, it is not surprising to find level correlations that belong to a different universality class. It would be very interesting to analyze the fate of the antiunitary symmetries and eigenvalue correlations in the continuum limit of the lattice gauge theory. Lattice simulations with larger values of β and bigger lattices are required to investigate this point.

For three or more colors the antiunitary symmetries are broken both in the continuum theory and for staggered lattice fermions. We predict that in this case the eigenvalue correlations are given by the GUE. It would be worthwhile to obtain the complete Dirac spectrum also for this case.

A final point of interest we want to mention is the fate of level correlations during the chiral phase transition. From solid state physics [15] we know a delocalization transition is associated with a transition in the level statistics which raises the hope that such phenomena can be seen in QCD as well.

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