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Linearly Positive Histories: Probabilities for a Robust Family of Sequences of Quantum Events

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Non-negative probabilities that obey the sum rules may be assigned to a much wider family of sets of histories than decohering histories. The resulting *linearly positive histories* avoid the highly restrictive decoherence conditions and yet give the same probabilities when those conditions apply. Thus linearly positive histories are a broad extension of decohering histories. Moreover, the resulting theory is manifestly time-reversal invariant.

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Recently there has been considerable interest in finding a formulation of quantum mechanics which yields for a closed system not only the probabilities of single events, but also the probabilities of sequences of events, or histories [1–8].

Single events may be described by projection operators P, which are Hermitian idempotent operators $P = P^\dagger = P^2$. When the quantum state of the closed system is given by the pure (for the moment) normalized positive semidefinite Hermitian density matrix $\rho = |\psi\rangle\langle\psi| = \rho^2$, the probability of the event is

$$p = \|P|\psi\rangle\|^2 = \langle\psi|P^\dagger P|\psi\rangle = \mathrm{Tr}(P\rho P^\dagger), \quad (1)$$

or, using the Hermitian idempotent property of P, the Hermiticity of ρ, and the cyclic property of the trace,

$$p = \langle\psi|P|\psi\rangle = \mathrm{Re}\langle\psi|P|\psi\rangle = \mathrm{Tr}(P\rho) = \mathrm{Tr}(\rho P^\dagger) = \mathrm{Re}\,\mathrm{Tr}(P\rho). \quad (2)$$

Any of these expressions written in terms of ρ rather than $|\psi\rangle$ may also be applied to a system in a mixed quantum state $\rho \neq \rho^2$ (but still Hermitian, positive semidefinite, and normalized, $\mathrm{Tr}\rho = 1$).

Although it is doubtful that they can be empirically tested in general [9], it is often desirable to be able to assign probabilities to sequences of events, which can be described by the class operator

$$C = P^{(n)}P^{(n-1)}\ldots P^{(2)}P^{(1)}, \quad (3)$$

or to more general histories, described by C's that are sums of such strings (3) of projection operators. Here the projection operators $P^{(i)}$ are projection operators at different times t_i, $t_1 < t_2 < \cdots < t_{n-1} < t_n$, and we are using the Heisenberg picture. When all of the projection operators commute, C itself is a projection operator, and Eqs. (1) and (2) would apply with P replaced with C. However, generically when the projection operators do not commute, C is not a projection operator. Then the different expressions in Eqs. (1) and (2) with P replaced by C may differ, and it becomes arbitrary which, if any, to use.

The usual choice [1–8] is to take the analog of Eq. (1) and, say,

$$p = \mathrm{Tr}(C\rho C^\dagger). \quad (4)$$

This choice has the positive feature that it is always non-negative. It is motivated by the fact that for single strings (3), this expression indeed gives the probability of observing the events $P^{(i)}$ in a sequence of ideal measurements on a system with an initial density matrix ρ. This is a routine consequence of applying the standard quantum formalism, including the collapse postulate, to

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the sequence of measurements. It also follows from regarding the measured system as a subsystem of a larger system, one in which, at successive times t_i, the subsystem is coupled to recording devices in such a way that a sequence of ideal quantum nondemolition measurements is performed, with the results stored in commuting records. Then either Eq. (1) or (2) applied to the total system, with the projection operator there being the product of the commuting projection operators for the records, gives the same answer as Eq. (4) applied to the subsystem treated as isolated.

However, we are now interested instead in assigning probabilities to a history of a closed system, such as the Universe, which is not being measured by any external device. In this case Eq. (4) cannot be derived from Eqs. (1) or (2) for a larger system. Instead, it must simply be postulated as a new formula.

A problem with Eq. (4) is that it generally does not give a set of probabilities obeying the standard sum rules. For an exhaustive set of histories $\{\alpha\}$, meaning that the sum of the corresponding class operators C_α is the identity

$$\sum_\alpha C_\alpha = I\,, \tag{5}$$

one can form a coarser grained set of histories $\{\hat{\alpha}\}$ by grouping together the α's into a set of a smaller number of exclusive and exhaustive $\hat{\alpha}$'s. The class operators for this coarser-grained set are obtained by summing

$$C_{\hat{\alpha}} = \sum_{\alpha \in \hat{\alpha}} C_\alpha\,, \tag{6}$$

and then Eq. (4) leads to a new set of probabilities

$$p_{\hat{\alpha}} = \mathrm{Tr}(C_{\hat{\alpha}} \rho C_{\hat{\alpha}}^\dagger)\,. \tag{7}$$

But now the trouble is that generically we do not have the probability sum rule

$$p_{\hat{\alpha}} = \sum_{\alpha \in \hat{\alpha}} p_\alpha\,. \tag{8}$$

A necessary and sufficient condition [1–3] that the sum rule (8) does hold for probabilities defined by Eqs. (4) and (7) is that

$$\mathrm{Re}\,\mathrm{Tr}(C_\alpha \rho C_{\alpha'}^\dagger) = 0 \text{ for all pairs } \alpha \neq \alpha'\,, \tag{9}$$

which is called the weak decoherence condition [3]. It is also closely related, but not identical, to the consistency condition [1,2] that was proposed earlier. In the consistent histories or decoherent histories approach to quantum mechanics [1–8], one only assigns probabilities, by Eq. (4), to consistent or decohering sets of histories $\{\alpha\}$ such that Eq. (9) or a slightly different version of it is exactly or approximately true. These probabilities then obey the usual rules, but they are only defined for highly restrictive sets of histories.

Here we propose a different new formula for the probabilities of histories in the quantum mechanics of a closed system, namely, the analog of the last expression of Eq. (2):

$$p_\alpha = \mathrm{Re}\langle\psi|C_\alpha|\psi\rangle = \mathrm{Re}\,\mathrm{Tr}(C_\alpha \rho)\,. \tag{10}$$

Because this is linear in C_α, it obviously obeys the probability sum rule (8) when

$$p_{\hat{\alpha}} = \mathrm{Re}\,\mathrm{Tr}(C_{\hat{\alpha}} \rho) \tag{11}$$

Therefore, these may be called *linear probabilities.*

The obvious problem with Eq. (10) is that it can be negative. Therefore, we impose the *linear positivity condition*

$$\mathrm{Re}\,\mathrm{Tr}(C_\alpha \rho) \geq 0 \tag{12}$$

for all $\alpha \in \{\alpha\}$. Such a set of histories $\{\alpha\}$ obeying the inequality (12) will be called a *linearly positive set of histories.* A member of such a set may then be called a *linearly positive history* (or *positive history* for short).

Because the linear positivity condition (12) for a given state ρ depends only on the C_α for each history in question, one can say whether or not a history is positive without also specifying in which set of histories it belongs. This is one immediate way in which the linear positivity condition is simpler than the weak decoherence condition (9), since the latter depends not only on the C_α of the history in question, but also on the $C_{\alpha'}$ of all other histories in the set. This dependence on the complete set of histories in the decohering case leads to the complication there of needing to consider the entire set before one can say whether any individual history is decohering, a complication that is entirely avoided for positive histories. (One could define an individual weakly decoherent history as one for which the minimal set, given by C_α and $C_{\alpha'} = I - C_\alpha$, is weakly decoherent. Every history in a weakly decoherent set of histories is an individual weakly decoherent history, but a complete set of more than two such individual weakly decoherent histories is generically not weakly decoherent, whereas any complete set of individually linearly positive histories is automatically positive.)

One can readily see that if the system is in a pure state, and if the class operator C_α is a product of rank-one projection operators onto a succession of pure states, $\mathrm{Tr}(C_\alpha \rho)$ is a product of transition amplitudes that start and end at the system state. If this product is nonzero, its phase is Berry's phase [10] for the closed circuit in the projective Hilbert space that follows the geodesic segments joining the successive states. Thus in this special case the linear positivity condition (12) is the condition that the corresponding Berry's phase is in the first or fourth quadrant (or at its edge).

Decoherent histories can be given a time-symmetric generalization motivated by [11] with both initial and final density matrices ρ_i and ρ_f (still Hermitian and positive semidefinite, but no longer necessarily normalized) [5]. Then the weak decoherence condition (9) becomes

$$\mathrm{Re}\,\mathrm{Tr}(\rho_f C_\alpha \rho_i C_{\alpha'}^\dagger) = 0 \text{ for all pairs } \alpha \neq \alpha', \qquad (13)$$

and the probabilities (4) become

$$p_\alpha = \mathrm{Tr}(\rho_f C_\alpha \rho_i C_\alpha^\dagger)/\mathrm{Tr}(\rho_f \rho_i). \qquad (14)$$

Similarly, the linear positivity condition (12) can be generalized to

$$\mathrm{Re}\,\mathrm{Tr}(\rho_f C_\alpha \rho_i) \geq 0, \qquad (15)$$

and the resulting linearly positive histories can be assigned the probabilities

$$p_\alpha = \mathrm{Re}\,\mathrm{Tr}(\rho_f C_\alpha \rho_i)/\mathrm{Tr}(\rho_f \rho_i). \qquad (16)$$

In either of these cases, single-state quantum mechanics of a closed system is the special case

$$\rho_f = c_1 I, \qquad \rho_i = c_2 \rho \qquad (17)$$

for any positive real constant numbers c_1 and c_2. Even if Eq. (17) does not hold, the linear positivity condition (15) and linear probabilities (16) for the two-state case are exactly the same as the analogous condition (12) and probabilities (10) if we take

$$\rho = \rho_i \rho_f/\mathrm{Tr}(\rho_i \rho_f), \qquad (18)$$

though this need not give a Hermitian density matrix ρ if the Hermitian ρ_i and ρ_f do not commute. Thus the two-state case is, in fact, a special case of an even broader generalization of linear positive histories, applying Eq. (10) to an arbitrary operator ρ that need not be Hermitian or positive semidefinite, though it should still be normalized so that the sum of the linear probabilities is unity.

Inserting (5) into (10) [or into (16)] and expanding, one finds [Eq. (14) of [6]] that for weakly decohering sets of histories, the probabilities assigned by Eqs. (4) and (10) [or by (14) and (16)] are identical. Since these probabilities are then all non-negative, we see that the weak decoherence condition implies the linear positivity condition and gives the same probabilities. Of course, the converse is not true.

Thus the set of all weakly decohering sets of histories is a proper subset of the set of all linearly positive sets of histories. In fact, the weak decoherence condition (9) or (13), being a set of *equations* (for all $\alpha \neq \alpha'$), is true only on a surface in the set of parameters describing a set of histories. On the other hand, the linear positivity conditions (12) or (15) are merely inequalities and so are true in a region (the closure of an open region) of the set of parameters describing a set of histories. That is, the set of all weakly decohering sets of histories is a subset of measure zero of the set of all linearly positive sets of histories, whereas the latter is a subset of positive measure in the set of all sets of histories.

In this way linearly positive histories are an enormous generalization of weakly decohering histories. The former enable one to assign sets of probabilities to a much broader family of sets of histories, avoiding the highly restrictive conditions (9) or (13) of weakly decohering histories. It is also obviously true that linearly positive histories are an even greater generalization of histories that obey the medium decoherence condition [3], which is Eq. (9) without Re on the left-hand side, or the strong decoherence condition [3], which is that there exists a complete set of orthogonal projection operators R_α such that

$$C_\alpha \rho = R_\alpha \rho. \qquad (19)$$

Because of the strong restrictions imposed by the equations for the various decoherence conditions, often these are loosened to approximate equalities [3,4]. However, this procedure has a certain vagueness or arbitrariness which is entirely avoided by the precise inequalities (12) or (15) of the linear positivity condition.

The linear probabilities (10) and the linear positivity condition (12) for the case of a single state ρ have the nice feature that they are automatically invariant under reversing the order of the projection operators in C_α, which replaces it by $C_\alpha^\dagger$. The same is true for (15) and (16) with both initial and final states ρ_i and ρ_f if they commute. Similarly, if we define the *CPT*-reversed history $\tilde{\alpha}$, represented by the class operator

$$\tilde{C}_\alpha = \Theta^{-1} C_\alpha^\dagger \Theta, \qquad (20)$$

which takes the *CPT* conjugates of the projection operators as well as reversing the order [5,6], then the linear probabilities and linear positivity condition are invariant under this "time reversal" in the one-state case if, as usual, ρ is replaced by its time reversal $\tilde{\rho} \equiv \Theta^{-1}\rho\Theta$ or, in the two-state case, if ρ_i and ρ_f are replaced, respectively, by $\tilde{\rho}_f$ and $\tilde{\rho}_i$. In particular, the linear probabilities and linear positivity condition are invariant (without any change of state) in the one-state case if ρ is *CPT* invariant or in the two-state case if

$$\rho_i \rho_f = \tilde{\rho}_f \tilde{\rho}_i \equiv \Theta^{-1}\rho_f \Theta \Theta^{-1} \rho_i \Theta, \qquad (21)$$

e.g., if $\rho_f = \tilde{\rho}_i \equiv \Theta^{-1}\rho_i\Theta$, or, alternatively, if ρ_i and ρ_f commute and are separately *CPT* invariant, i.e., if $[\rho_i, \rho_f] = 0$, $\rho_i = \tilde{\rho}_i$, and $\rho_f = \tilde{\rho}_f \equiv \Theta^{-1}\rho_f\Theta$.

It is perhaps worth emphasizing that a set of histories defining a sequence of *measurements* automatically satisfies not merely the linear positivity condition (12) but also the weak decoherence condition (9), when the formulas are applied to the records of the measurements. (This is true because each C_α is then a product of projection operators that commute, namely, one projection operator for each independent record of the corresponding measurement.) Thus the formulas (10) and (4) agree in this case.

In the case of *ideal* measurements, we could as well have considered the measured system projections with which the records are correlated. Moreover, in this case one gets the same probability from Eq. (4) even if one replaces the

C_α that is the product of commuting projection operators onto the records with the C_α that is the product of the corresponding (generically noncommuting) projection operators onto the measured system *treated as closed,* i.e., with the Heisenberg projections defined in terms of the unitary evolution of this system in isolation. (This is why in ideal cases one can correctly calculate the probabilities by an analysis of the measured system alone, ignoring the quantum mechanics of the measuring apparatus.) However, if this replacement is made for each C_α, then Eq. (10) does *not* generically give the same answer as Eq. (4), even when the histories are still linearly positive. Thus the probabilities of linearly positive histories depend crucially on what measurements are actually made.

For example, consider a spin-$\frac{1}{2}$ system with $\rho = |\sigma_z = 1\rangle\langle\sigma_z = 1|$, $P_1^{(1)} = |\sigma_x = 1\rangle\langle\sigma_x = 1|$, $P_1^{(2)} = |\sigma_z = 1\rangle\langle\sigma_z = 1|$, and $P_2^{(i)} = I - P_1^{(i)}$. The corresponding set of histories, with elementary class operators (using time-ordered labeling) $C_{11} = P_1^{(2)}P_1^{(1)}$, $C_{21} = P_1^{(2)}P_2^{(1)}$, $C_{12} = P_2^{(2)}P_1^{(1)}$, $C_{22} = P_2^{(2)}P_2^{(1)}$, is not weakly decoherent—because of the obvious interference—but it *is* linearly positive, with probabilities $p_{11} = p_{21} = 1/2$, $p_{12} = p_{22} = 0$. However, if a measurement of the first spin (i.e., of σ_x at time t_1) is incorporated into the histories, the resulting set of histories *will* be weakly decoherent, and the probabilities will all become 1/4.

We note finally that for a different category of histories than the category [of all histories of the form (3)] considered in this paper, namely, the category of histories each of which is given by a collection of (fine-grained) trajectories in configuration space alone, it is possible to extend the formula (1) applied to configurational events at any single time to a probability distribution on the set of *all* possible configurational trajectories [12]. One may thus wonder whether an even broader extension [of (4) applied to weakly decohering histories] than that provided by (10) applied to linearly positive histories is possible, an extension which consistently assigns probabilities to *all* possible histories (3). We note in this regard that such an extension is precluded by the usual no-hidden-variables theorems [13]. [These theorems show, in fact, much more: that it is even impossible to have an extension, to all histories, of (4) restricted to histories for which the projections in the sequence (3) mutually commute.] In other words, the totality of different weakly decohering sets of histories, or of different linearly positive sets of histories, with their respective probability formulas, is genuinely inconsistent—in the sense that the "probability" assignments for these different sets of histories cannot simultaneously be realized as relative frequencies within a single ensemble. This shows, in fact, that whatever may be the virtues of the linear positivity condition, within the framework considered here it cannot eliminate the necessity, emphasized by Gell-Mann and Hartle [3,4], of formulating additional conditions on sets of histories which select from this totality a limited number of sets of histories, and perhaps a unique set of histories (e.g., one which defines what Gell-Mann and Hartle call the "quasiclassical domain of familiar experience" [3]).

To summarize, linear probabilities (10) or (16) may be applied to a much broader class of histories than weakly decohering histories. That is, they may be applied to our proposed linearly positive sets of histories, which are sets of histories obeying the linear positivity condition (12) or (15), namely, the condition that the linear probabilities are all non-negative. These linear probabilities obey the sum rules and are equal to the previously proposed probabilities (4) or (14) in the very special subset of cases obeying the weak decoherence condition (9) or (13) necessary for the probabilities (4) or (14) also to obey the sum rules.

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