Fixed-Point Structure of Scalar Fields

Kenneth Halpern and Kerson Huang

Department of Physics and Center for Theoretical Physics, Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

We search for alternatives to the trivial ϕ^4 field theory by considering nonpolynomial potentials. Such theories are renormalizable when the natural cutoff dependences of the coupling constants are taken into account. We find a continuum of fixed points, which includes the well-known Gaussian fixed point. The fixed-point density has a maximum at a location corresponding to a theory with a Higgs boson mass of approximately 2700 GeV. The Gaussian fixed point is UV stable in some directions in the extended parameter space. Along such directions we obtain nontrivial asymptotically free theories.

PACS numbers: 11.10.Hi, 11.10.Kk, 11.10.Lm

The Higgs field of the standard model, usually taken to be a scalar field with quartic self-interaction, has been overshadowed by the issue of "triviality"; namely, the renormalized coupling vanishes in the limit of infinite cutoff [1]. This has been verified by numerical calculations [2], and the implications for phenomenology have been examined [3]. In terms of Wilsons's renormalization group (RG) [1], the reason for triviality is that the continuum (or infinite-cutoff) limit is identified with an infrared (IR) Gaussian fixed point, so that the theory approaches a free field theory in the low-energy limit. It is natural to ask whether there are alternative continuum limits that yield a nontrivial theory. As one of us [4] noted earlier, in a ϕ^4 theory in $4 - \epsilon$ dimensions there are RG trajectories on which the Gaussian fixed point appears as an UV fixed point. Theories built along these trajectories would be nontrivial and asymptotically free. Unfortunately, they become trivial as $\epsilon \to 0$. However, this scenario has led us to search for nontrivial theories in an extended parameter space. In this Letter we report on some positive results.

The model being investigated is an N-component real scalar field theory in d dimensions, with arbitrary power-law self-coupling. We are interested in how the couplings transform under a change of energy scale, and the inclusion of all powers is necessary for closure under the RG. The theory remains renormalizable in the usual perturbative sense when we recognize that the coupling constants must depend on the cutoff in specific manners. These dependences are such that the S matrix of the theory for $d = 4$ is that of an effective ϕ^4 theory, whose effective coupling depends on all the coupling constants of the underlying theory. To study the renormalization of the effective coupling, we must examine the RG flow in the infinite-dimensional parameter space of the extended theory. This is done using Wilson's method of momentum-shell integration, and the main results may be summarized as follows.

(a) In the extended parameter space, some RG trajectories flow into the Gaussian fixed point, while others flow out of it. The Gaussian fixed point is UV stable with respect to the latter type of trajectories, along which the theory is nontrivial and asymptotically free. Spontaneous symmetry breaking occurs along some trajectories of this type.

(b) There exists a one-parameter continuum of nontrivial fixed points. For the Higgs field ($N = 4$, $d = 4$) the density of fixed points is maximum at a location corresponding to a potential with broken symmetry. The Higgs boson mass calculated from this potential is approximately 2700 GeV.

The model.—The Euclidean action of our model is

$$S[\phi] = \int d^d x \left[\frac{1}{2} (\partial \phi_i)^2 + V(\phi^2) \right],$$

$$V(\phi^2) = \sum_{n=1}^{\infty} g_{2n} (\phi^2)^n, \tag{1}$$

where $\phi_i(x)$ ($i = 1, \ldots, N$) are real fields, and $\phi^2 = \phi_i \phi_i$. There is a momentum cutoff (or equivalent inverse lattice spacing) Λ, which is assumed to be the only intrinsic scale in the theory. Since the coupling constant g_K has dimension $(\text{momentum})^{K+d-Kd/2}$, it must depend on the cutoff according to

$$g_K = u_K \Lambda^{K+d-Kd/2}, \tag{2}$$

where u_K is a dimensionless parameter. We define $u_2 = r/2$, and write for the bare-mass square

$$m_0^2 \equiv 2g_2 = r\Lambda^2. \tag{3}$$

In the conventional renormalization scheme, a dimensionless coupling constant is considered renormalizable; but any higher coupling is rejected on the basis that it leads to divergencies that cannot be canceled by a finite number of counter terms. In d dimensions, according to this rule, only ϕ^M and lower-order theories are renormalizable, where $M = 2d/(d - 2)$. With the cutoff dependence (2) taken into account, however, the higher couplings do not

generate new divergencies, and are renormalizable. The reason is that they vanish in the limit $\Lambda \to \infty$, thus supplying extra convergent factors to Feynman graphs [5]. A more detailed analysis, supplementing the usual power counting [6] with the cutoff dependences from (2), shows that, for $d \geq 4$, a higher vertex can contribute to a skeletal graph only if the graph has M or fewer external lines. For a nonskeletal graph with more than M external legs, higher vertices can contribute only via vertex insertions. This means that the S matrix of the theory is that of an effective ϕ^M theory, whose effective coupling depends on all the dimensionless couplings u_K. Thus, renormalization must be discussed in the context of the underlying theory. In particular, the fixed points of the effective coupling are projections of fixed points in the extended parameter space, and, except for the Gaussian fixed point, cannot be found from the effective theory.

Renormalization-group equations.—We make use of Wilson's RG transformation [1], which thins out degrees of freedom by (a) integrating out Fourier components of ϕ_i with momenta lying in a shell between Λ and Λ/b, (b) rescaling the cutoff back to Λ, and (c) rescaling $\phi(x)$ to restore normalization in (1). We put

$$t = \ln b, \tag{4}$$

so that $t = 0$ corresponds to the energy scale at the cutoff. The transformation is effected by first decomposing the field into two terms $\phi = \phi_s + \phi_f$, where ϕ_s (the slow piece) has nonvanishing Fourier components only for momenta k with $|k| < \Lambda/b$, while ϕ_f (the fast piece) has $\Lambda/b \leq |k| \leq \Lambda$. The action is split into two terms, the free-field (quadratic) action $S_0[\phi]$, which includes the kinetic and the bare-mass term, and $S_1[\phi]$, which contains the remaining terms. We can write $S[\phi_s + \phi_f] = S_0[\phi_s] + S_0[\phi_f] + S_1[\phi_s + \phi_f]$, and put the partition function in the form

$$Z = \int (D\phi_s) \exp\{-S_0[\phi_s]\}$$

$$\times \int (D\phi_f) \exp\{-S_0[\phi_f] - S_1[\phi_s + \phi_f]\}$$

$$\equiv \int (D\phi_s) \exp\{-S'[\phi_s]\}. \tag{5}$$

We then rescale all momenta by a factor of b to restore the apparent cutoff to Λ, rescale ϕ_s to restore our former normalization, and read off the new coupling constants from the rescaled S'. The RG trajectories in the parameter space are generated through successive RG transformations.

In practice, the RG transformation is carried out in perturbation theory by expanding the partition function in powers of $S_1[\phi_s + \phi_f]$, where ϕ_f is the field to be integrated over, with ϕ_s held fixed. In the Feynman graphs of this expansion, all internal lines correspond to ϕ_f, and external lines correspond to ϕ_s. The renormalized value of u_K has contributions from connected graphs with K external lines.

We calculate only to first order in t, with an infinitesimally thin momentum shell. This yields $\partial u_K / \partial t$ at $t = 0$, or β functions at $t = 0$. For this calculation we only need to include tree and one-loop graphs, because the momenta of internal lines have only an infinitesimal range, and any additional loop integration will vanish in the limit $t \to 0$, because its range goes to zero. In general, the RG transformation generates all powers of the self-coupling, even if they were not present in the beginning. In addition, derivative couplings will arise, but are ignored in the current analysis.

The continuum limit $\Lambda \to \infty$ is not trivial, because Λ is invisible: In the action, Λ can be absorbed through a rescaling of the field. To find the value of Λ, we have to compute some physical quantity, for example, a correlation length, which will be given in units of Λ^{-1} (or equivalent lattice spacing). We only know that $\Lambda \to \infty$ when the correlation length diverges, and this can be true only at certain RG fixed points. A continuum limit therefore corresponds to some fixed point, and different fixed points define different physical theories.

As mentioned above, the β functions are calculated at $t = 0$ (the energy scale of the cutoff) by summing one-loop graphs. But, since Λ does not explicitly appear in the action, these functions are characterized solely by the values of the u_K. Thus, the β functions are functions only of the u_K, and our one-loop RG equations are exact except for the neglect of derivative coupling terms. They are as follows:

$$\frac{\partial u_{2n}}{\partial t} = (2n + d - nd)u_{2n} + \frac{S_d}{2} \sum_{k=1}^{n} \frac{(-1)^{k-1} 2^k}{k(1 + r)^k}$$

$$\times \sum_{\{m_1,\ldots,m_k\}}' P(m_1,\ldots,m_k) u_{2m_1} \cdots u_{2m_k}$$

$$(n = 1,\ldots,\infty), \tag{6}$$

where $u_K = u_K(t)$, and

$$P(m_1,\ldots,m_k) = (m_1 \cdots m_k)\{[(2m_1 - 1) \cdots (2m_k - 1)]$$

$$+ (N - 1)\}. \tag{7}$$

In (6), r is the bare-mass parameter defined in (3), and $S_d = 2^{1-d} \pi^{-d/2}/\Gamma(d/2)$ is the surface area of a unit d-sphere divided by $(2\pi)^d$, with $S_4 = (8\pi^2)^{-1}$. In the sum over $\{m_1,\ldots,m_k\}$, the prime indicates the conditions

$$m_i = 2,\ldots,n - k + 2,$$

$$\sum_{i=1}^{k} m_i = n + k. \tag{8}$$

On the right side of (6), the first term comes from rescaling. The second term contains the one-loop contributions, in which k is the number of vertices on the loop, with respective orders $2m_1,\ldots,2m_k$.

The RG equations (6) are implicit in the work of Wegner and Houghton [7], who used slightly different methods. Using the Wegner-Houghton results, Hasenfratz and Hasenfratz [8] analyzed effectively the same equations as

(6), and concluded that there are no nontrivial fixed points. As we shall see, this is true when the potential is restricted to be a polynomial; but interesting nontrivial results obtain for nonpolynomial potentials (exponentially rising ones, as it turns out).

Gaussian fixed point.—An obvious fixed point of the RG equations is the Gaussian fixed point, with all $u_{2n} = 0$. The directions of stability and instability can be found by diagonalizing the linearized RG equations. A negative eigenvalue corresponds to an IR-stable direction, while a positive one corresponds to an UV-stable direction.

We quote the results for the case $d = 4$, $N = 4$, which corresponds to the Higgs field. For the eigenvalue

$$\lambda = 2(a + 1), \tag{9}$$

the components of the eigenvector are given by

$$u_{2n}(0) = \frac{r(0)\,(8\pi^2)^{n-1}[(a + 1)\cdots(a + n - 1)]}{n!(n + 1)!}$$

$$(n = 1, \ldots, \infty). \tag{10}$$

The Gaussian fixed point is IR stable for $a < -1$, UV stable for $a > -1$, and marginal for $a = -1$. The eigenpotential defined by $V_a(\phi^2, t) \equiv \sum_{n=1}^{\infty} u_{2n}(t)\,(\phi^2)^n$ has the property

$$(\partial V_a/\partial t)_{t=0} = \lambda V_a, \tag{11}$$

and is given at $t = 0$ by

$$V_a(\phi^2, 0) = \frac{r(0)}{8\pi^2 a}[M(a, 2, 8\pi^2\phi^2) - 1], \tag{12}$$

where $M(a, b, z)$ is a Kummer function [9]

$$M(a, b, z) = \sum_{n=0}^{\infty} \frac{a(a + 1)\cdots(a + n - 1)}{b(b + 1)\cdots(b + n - 1)}\frac{z^n}{n!}$$

$$= \frac{\Gamma(b)}{\Gamma(b - a)\Gamma(a)}\int_0^1 dt\, e^{zt}t^{a-1}(1 - t)^{b-a-1}. \tag{13}$$

The series breaks off to become a polynomial for negative integer values $a = -1, -2, \ldots$. The case $a = -1$, corresponding to the free field, is marginal (has eigenvalue zero). The cases $a = -2, -3, \ldots$ are IR stable. This shows that all polynomial interactions become irrelevant in the low-energy limit, and the higher the polynomial degree the greater the irrelevancy. This justifies the neglect of higher terms than ϕ^4 in the conventional Higgs sector. There also exist other IR-stable cases corresponding to nonpolynomial interactions with noninteger $a < -1$.

All the positive eigenvalues, with $a > -1$, correspond to nonpolynomial interactions. The eigenpotentials in these cases rise like $z^{a-2}e^z$ for large $z \equiv 8\pi^2\phi^2$, and thus have different norms from the polynomial potentials. They describe nontrivial asymptotically free theories. The "running" potential $V_a(\phi^2, t)$ can be found by solving the nonlinear RG equations with given initial conditions. Initial potentials with $0 < \lambda < 2$ and $r(0) < 0$ yield theories with asymptotic freedom and spontaneous symmetry breaking. Sufficiently closed to the Gaussian fixed point,

where the linear approximation holds, the eigenpotentials corresponding to these conditions have one and only one minimum. An approximate calculation of properties near the minimum gives the following estimate for the squared ratio of Higgs boson mass to vacuum field:

$$\frac{m^2}{\langle\phi^2\rangle} \approx \frac{8\pi^2}{3}\sqrt{2\lambda(1 + \lambda)}[-r(t)]. \tag{14}$$

Nontrivial fixed points.—To find all the fixed points of our theory, we set $\partial u_{2n}/\partial t = 0$ in (6), thereby obtaining a recursion formula for the u_{2n} at the fixed point, with one free parameter r. Therefore, we have a one-dimensional continuum of fixed points—a fixed line—with the Gaussian fixed point located at $r = 0$. Figure 1 shows a plot of u_{2n} as a function of r for $n = 2, \ldots, 30$, for the case $d = 4$, $N = 4$. They are normalized to 1 at the maxima. With the exception of the case $n = 2$, all maxima occur at approximately the same point $r_{\max} = -0.65$. This indicates that the density of fixed points has a maximum at $r = r_{\max}$. The potential at this point exhibits spontaneous symmetry breaking, and yields a ratio of Higgs boson mass to vacuum field of approximately 10. Using a vacuum field of 273 GeV gives a Higgs boson mass of approximately 2700 GeV. The significance of the accumulation point on the fixed line is yet unclear, and merits further study.

The novel features we found in $d = 4$, namely, asymptotic freedom from the Gaussian fixed point, and the existence of nontrivial fixed points, are associated with potentials that have the asymptotic form $\exp(c\,\phi^2)$, whereas Hasenfratz and Hasenfratz [7] restricted their potential to be polynomial by implicit assumption. We verify their finding that no nontrivial fixed points exist with polynomial potentials. There is, however, no mathematical or physical reason why exponential potentials should be excluded.
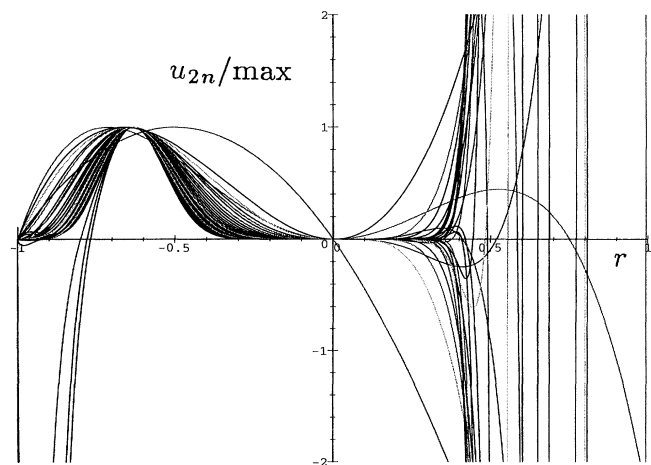

FIG. 1. Plot of the coefficients u_{2n} ($n = 2, \ldots, 30$) along the fixed line parametrized by r, normalized to 1 at the maximum of each curve. With one notable exception ($n = 2$) all the maxima approximately coincide at $r = -0.65$.

The much-studied two-dimensional Liouville theory [10], for example, is based on an exponential potential.

The triviality of the conventional approach furnishes the incentive for us to explore new directions, and the preliminary results reported here indicate that such ventures may not be altogether fruitless. A more thorough paper will be published separately, where we hope to discuss unresolved issues such as the effect of higher derivative couplings.

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