Why the Entropy of a Black Hole is $A/4$

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The paper analyzes the relation between Bekenstein-Hawking entropy S^{BH} of a black hole and the statistical-mechanical entropy S^{SM} defined by counting its internal degrees of freedom. The mechanism explaining universality of S^{BH} is proposed.

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According to the thermodynamical analogy in black hole physics, the entropy of a black hole in the Einstein theory of gravity equals $S^{\mathrm{BH}} = A_H/(4l_{\mathrm{P}}^2)$, where A_H is the area of a black hole surface and $l_{\mathrm{P}} = (\hbar G/c^3)^{1/2}$ is the Planck length [1,2]. This entropy plays essentially the same role as in the usual thermodynamics. In particular, it defines the response of the free energy F of a system containing a black hole on the change of the temperature T: $dF = -S^{\mathrm{BH}}\, dT$. The calculations in the framework of the Euclidean approach initiated by Gibbons and Hawking [3,4] relate this quantity with the tree-level contribution of the gravitational action, namely, the action of the Euclidean black hole instanton. In this approach it remains unclear whether there exist any real dynamical degrees of freedom which are responsible for it.

The problem of the dynamical origin of the black hole entropy was intensively discussed recently (see, e.g., [5–11]). The basic idea which was proposed is to relate the dynamical degrees of freedom of a black hole to its quantum excitations. This idea has different realizations. In particular, it was proposed to identify the dynamical degrees of freedom of a black hole with the states of all fields (including the gravitational one) which are located inside the black hole [6,11]. By averaging over states located outside the black hole, one generates the density matrix $\hat{\rho}$ of a black hole and can calculate the corresponding statistical-mechanical entropy of a black hole $S^{\mathrm{SM}} = -\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})$. The main contribution to the entropy is given by modes of fields located inside the black hole in the very close vicinity of the horizon. The generic feature of this as well as other "dynamic" approaches is that the statistical-mechanical entropy of a black hole arises at the one-loop level. However, the relation between the "topological" (tree-level) calculations and one-loop calculations based on the counting dynamical degrees of freedom of a black hole remains unclear. In particular, S^{SM} depends on the number and characteristics of the fields, while S^{BH} does not. What is the relation between the Bekenstein-Hawking entropy S^{BH} and statistical-mechanical entropy S^{SM}, and what is the mechanism which provides universality of S^{BH}? The aim of this Letter is to clarify these questions.

First of all we note that thermodynamical and statistical-mechanical definitions of entropy are logically different. In principle, the thermodynamical entropy S^{TD}, defined by the resonse of the free energy on the change of the temperature may differ from S^{SM}. Consider, for example, a thermodynamical system with a Hamiltonian $\hat{H}$. The variation of its free energy is [12]

$$dF = -S^{\mathrm{SM}}\, dT + \sum_{i=1}^{n} f^i\, d\lambda_i\,. \tag{1}$$

Here S^{SM} is the statistical-mechanical entropy $S^{\mathrm{SM}} = -\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})$, $\hat{\rho} = \rho_0 \exp(-\hat{H}/kT)$, $f^i = \overline{(\partial\hat{H}/\partial\lambda)}\, d\lambda_i$, and λ_i are parameters on which the Hamiltonian $\hat{H}$ depends. If in a state of thermal equilibrium the part of these parameters (λ_a, $a = 1,\ldots,m \le n$) is uniquely defined by the temperature T one has

$$dF = -S^{\mathrm{TD}}\, dT + \sum_{i=m+1}^{n} f^i\, d\lambda_i\,, \tag{2}$$

where $S^{\mathrm{TD}} = S^{\mathrm{SM}} + \Delta S$, and $\Delta S = \sum_{a=1}^{m} f^i (d\lambda_i/dT)$. For such a system the thermodynamical entropy S^{TD} determined by the total response of free energy on the change of temperature differs from S^{SM} obtained by counting the dynamical degrees of freedom.

We show that for a black hole $S^{\mathrm{TD}} \neq S^{\mathrm{SM}}$. It happens because characteristics of a nonrotating black hole considered as part of a thermodynamcal system are determined only by one parameter (mass M), which in turn in a state of a thermal equilibrium is uniquely defined by the temperature of the system. By varying the temperature one at the same time inevitably changes the background geometry in which the dynamical degrees of freedom are propagating. As the result the statistical-mechanical entropy S^{SM}, obtained by summing the contributions to the entropy of all internal degrees of freedom of the black hole and defined as $S^{\mathrm{SM}} = -\mathrm{Tr}(\hat{\rho}\ln\hat{\rho})$, does not coincide with the thermodynamical entropy S^{TD} of a black hole. Moreover, the additional terms, which arise due to the dependence of the geometry of a black hole and the number of states of quantum fields inside it on the temperature, exactly compensate the dominant statistical-mechanical contribution S^{SM} of the quantum field to the entropy. As a result of this compensation, the thermodynamical entropy of a black hole coincides with the Bekenstein-Hawking entropy S^{BH}, determined by the dependence of the mass (and hence the energy) of a black hole on the temperature (i.e., by the tree-level free energy F_0).

In order to derive the thermodynamical characteristics of a black hole it is convenient to begin with the partition function $Z(\beta)$. It is related with the free-energy F $[Z(\beta) = \exp(-\beta F)]$ and is defined by the functional integral [4,13]

$$Z(\beta) = \int D[g, \phi] \exp(iI[g, \phi]), \tag{3}$$

where $I[g, \phi]$ is the action for the gravitational field g and some other fields ϕ. The state of the system is determined by the choice of the boundary conditions on the metrics and fields that one integrates over. For the canonical ensemble describing the gravitational fields within a spherical box of radius r_B at temperature T_B one must integrate over all the metrics inside r_B which are periodically identified in the imaginary time direction with period $\beta_B = T_B^{-1}$. Denote by (g_0, ϕ_0) a point of the extremum of the action $I[g, \phi]$, then

$$\ln Z = iI[g_0, \phi_0] + \ln \int D[\bar{g}, \bar{\phi}] \exp(iI_2[\bar{g}, \bar{\phi}]), \tag{4}$$

where $\bar{g}_{\mu\nu} = g_{\mu\nu} - g_{0\mu\nu}$, $\bar{\phi} = \phi - \phi_0$, and $I_2[\bar{g}, \bar{\phi}] = I[g, \phi] - I[g_0, \phi_0]$ is quadratic in the perturbations $\bar{g}$ and $\bar{\phi}$. For vanishing background field $\phi_0 = 0$, the extremum g_0 is a solution of the vacuum Einstein equations, which for given boundary conditions coincides with the Euclidean black hole (a Hawking-Gibbons instanton), while $iI[g_0, \phi_0] = -I_E[g_0]$, where I_E is the Euclidean action. The relation (8) implies that the free-energy F can be written as

$$F = F_0 + F_1 + \cdots, \tag{5}$$

where $F_0 = \beta^{-1} I_E[g_0]$ and F_1 are the tree-level and one-loop contributions, respectively, and dots denote higher order terms in loops expansion.

A black hole will be in equilibrium with the thermal radiation inside the cavity if its mass M is related to β as $\beta = 4\pi r_+$, where $r_+ = 2M$ is the gravitational radius. (We use units in which $G = c = \hbar = 1$.) The equilibrium is stable if $r_B < 3r_+/2$. The tree-level contribution of the black hole to the free energy of the system can be written in the form [14,15]

$$F_0 = r_B\left(1 - \sqrt{1 - r_+/r_B}\right) - \pi r_+^2 \beta_B^{-1}, \tag{6}$$

where $\beta_B = 4\pi r_+ (1 - r_+/r_B)^{1/2}$ is the inverse temperature at the boundary r_B. The tree-level contribution S_0^{TD} to the thermodynamical entropy of a black hole defined as $S_0^{\mathrm{TD}} = -\partial_{T_B} F_0 \equiv \beta_B^2 \partial_{\beta_B} F_0$ is $S_0^{\mathrm{TD}} = \pi r_+^2 \equiv A/4l_{\mathrm{P}}^2 \equiv S^{\mathrm{BH}}$. In addition to this tree-level contribution, which identically coincides with the Bekenstein-Hawking entropy S^{BH}, there are also one-loop contributions directly connected with the dynamical degrees of freedom of the black hole describing its quantum excitations. We consider them now on more details.

By using Eq. (4) the one-loop contribution F_1 to the free energy can be written in the from $F_1 = -\beta^{-1} \ln Z_1$, where

$$Z_1 = \int D[\varphi] \exp(-I_E[\varphi]), \tag{7}$$

and $I_E[\varphi]$ is the quadratic Euclidean action of the field configuration $\varphi \equiv (\bar{g}, \bar{\phi})$. The integration is performed over all the perturbations fields φ that are real on the Euclidean section with metric

$$ds^2 = B d\tau^2 + B^{-1} dr^2 + r^2 d\Omega^2, \tag{8}$$

$B = 1 - r_+/r$ ($d\Omega^2$ is a line element on the unit sphere), and are periodic in imaginary time coordinate τ with period β [16]. In the one-loop approximation, different fields give independent contributions to F_1. For this reason it is sufficient to calculate the contribution of a chosen field φ and then add all the contributions corresponding to different fields. The integral (7) is ultraviolet divergent and requires regularization. The regularized value of F_1 may depend on some regularization mass parameter μ [17,18]. Below we assume that the corresponding regularization is made and use the notation Z_1 and F_1 for the renormalized values of these quantities [19].

According to its definition the one-loop contribution S_1^{TD} to the thermodynamic entropy of a black hole is determined by the total response of the one-loop free-energy F_1 on the change of the temperature. Besides the direct dependence of F_1 on temperature it also depends on the mass M of a black hole. In the thermal equilibrium M is a function of temperature. Thus we have ($r_+ = 2M$):

$$S_1^{TD} = \beta^2 \frac{dF_1}{d\beta} \equiv \beta^2 \frac{\partial F_1}{\partial \beta}\bigg|_{r_+} + \beta^2 \frac{\partial F_1}{\partial r_+}\bigg|_{\beta} \frac{dr_+}{d\beta}. \tag{9}$$

The first term in the right-hand-side of this relation is equal to the one-loop contribution S_1^{SM} to the statistical-mechanical entropy. In order to justify this claim we use the fact that the partition function Z_1 is related to the thermodynamical partition function $Z^T(\beta)$ of the canonical ensemble

$$Z^T(\beta) = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum \exp(-\beta E_n), \tag{10}$$

where E_n is the energy (eigenvalue of the Hamiltonian $\hat{H}$ of the field φ). Namely, Allen [17] showed that $F_1 \equiv -\beta^{-1} \ln Z_1$ differs from $F^T \equiv -\beta^{-1} \ln Z^T$ only by terms which are independent of β. Hence we have $\beta^2 (\partial F_1/\partial\beta)_{r_+} = \beta^2 (\partial F^T/\partial\beta)_{r_+} = -\mathrm{Tr}(\hat{\rho} \ln \hat{\rho}) \equiv S_1^{SM}$, where $\hat{\rho} = \exp[-\beta(\hat{H} - F^T)]$. (Here we used that the partial derivative $(\partial/\partial\beta)_{r_+}$ with respect to the inverse temperature β commutes with the Tr operation.) The above relations allow one to rewrite Eq. (9) in the form

$$S_1^{\mathrm{TD}} = S_1^{\mathrm{SM}} + \Delta S_1, \tag{11}$$

where $\Delta S_1 \equiv \beta^2 (\partial F_1/\partial r_+)_\beta \, dr_+/d\beta$. This relation shows that in order to obtain S_1^{TD} the statistical-mechanical entropy must be "renormalized" by adding ΔS_1. In particular, the relation (11) may give an explanation to the entropy renormalization procedure proposed by Thorne and Zurek [20].

For an investigation of ΔS_1, it is convenient to rewrite F_1, which enters the definition of ΔS_1 as $F_1 = (F_1 - F^T) + F^T$. The difference $F_1 - F^T$ does not depend on β and hence one can calculate its value for zero temperature ($\beta = \infty$). It indicates that the corresponding contribution to ΔS_1 is directly connected with vacuum polarization. The second contribution to ΔS_1 (connected with F^T term) arises because the complete derivative $d/d\beta$, defined by the relation (9), does not commute with Tr operation [21]. It is instructive to demonstrate in more detail the origin of this noncommutation. The thermodynamical partition function Z^T in a static spacetime can be presented in the form [17]

$$Z^T = \prod_\lambda [1 - \exp(-\beta\omega_\lambda)]^{-1}, \qquad (12)$$

where ω_λ are the energies of the single-particle states (or modes), and λ is the index enumerating these states. For the free-energy F^T we have

$$F^T = \sum_\lambda f(\beta\omega_\lambda) = \int d\omega\, N(\omega|r_+) f(\beta\omega), \qquad (13)$$

where $f(\beta\omega) = \beta^{-1}\ln[1 - \exp(-\beta\omega)]$, and $N(\omega|r_+)$ is the density of number of states at the given energy ω in a black hole of mass $M = r_+/2$. $dN/d\beta \neq 0$, since $N(\omega|r_+)$ depends on the mass of a black hole [22]. This implies that $d/d\beta$ and the Tr operation do not commute.

The calculation of the quantities which enter Eq. (11) is quite complicated. But important conclusions can be easily obtained by using some general properties of the free-energy F_1. In general case (if $\beta \neq \beta_H = 4\pi r_+$) the free-energy F_1 contains a divergence connected with the space integration over the region near the horizon. In order to regularize this divergence we suppose that the integration is performed up to the proper distance l to the horizon. Denote $\varepsilon = (l/r_+)^2$. In order to emphasize the dependence of F_1 on the dimensionless cutoff parameter ε we shall write $F_1 = F_1(\beta, r_+, \varepsilon)$. The free energy has the same dimensions as r_+^{-1} and hence it can be presented in the form $F_1(\beta, r_+, \varepsilon) \equiv r_+^{-1}\mathcal{F}(\beta/\beta_H, \varepsilon)$, where $\mathcal{F}$ is dimensionless function of two dimensionless variables and $\beta_H = 4\pi r_+ = 1/T_H$ (T_H is the black hole temperature). The structure of the divergence near the Euclidean horizon can be analyzed by using the curvature expansion of F_1. The leading divergence near the horizon $r = r_+$ term is

$$\mathcal{F}(\beta/\beta_H, \varepsilon) \approx -\varepsilon^{-1} f(\beta/\beta_H). \qquad (14)$$

An explicit form of the function f can be obtained by analyzing the free energy in a flat cone space. The high-temperature expansion [23] (see, e.g., Dowker and Kennedy [18]) shows that $f(x) \sim x^{-4}$ for $x \to \infty$. The divergence of F_1 at the Euclidean horizon for $\beta \neq \beta_H$ reflects the fact that the number of modes that contribute to the free energy and entropy is infinitely growing as one considers regions closer and closer to the horizon [6]. For $\beta = \beta_H$, the metric (8) is regular at the Euclidean horizon, and, hence, the renormalized free energy calculated for the regular Euclidean manifold is finite. It implies that $f(1) = 0$.

The one-loop contribution of a quantum field φ to the statistical-mechanical entropy is

$$S_1^{\mathrm{SM}} = \left[\beta^2 \frac{\partial F_1}{\partial \beta}\right]_{\beta=\beta_H}. \qquad (15)$$

It should be stressed that one must put $\beta = \beta_H$ only after the differentiation. The leading (divergent near the horizon) term of S_1^{SM} is

$$S_1^{\mathrm{SM}} \approx -\frac{4\pi}{\varepsilon} f'(1). \qquad (16)$$

For a conformal massless scalar field, $f'(1) = -1/(360\pi)$, and the expression (16) reproduces the result obtained in Ref. [6]. If the proper-distance cutoff parameter l is of the order of the Planck length l_P then the contribution of the field to the statistical-mechanical entropy of a black hole is of the order $S_1^{\mathrm{SM}} \sim A/l_P^2$, where A is the surface area of the black hole. In other words, for the "natural" choice of the cutoff parameter $l \sim l_P$ the one-loop statistical-mechanical entropy S_1^{SM} of a black hole is of the same order of magnitude as the tree-level Bekenstein-Hawking entropy S^{BH}.

Consider now one-loop contribution S_1^{TD} to the thermodynamical entropy of the black hole. According to Eq. (9) S_1^{TD} can be obtained by differentiation of F_1 with respect to the inverse temperature, provided one substitutes $\beta_H = \beta$ into F_1 *before* its differentiation

$$S_1^{\mathrm{TD}} = 4\pi\beta_H^2 \frac{\partial}{\partial\beta_H}\left(\frac{\mathcal{F}(1,\varepsilon)}{\beta_H}\right)_{\varepsilon=0}. \qquad (17)$$

Because for $\beta = \beta_H$ the free energy F_1 does not contain divergence at the Euclidean horizon, S_1^{TD} is finite. It means that the additional contribution ΔS_1 exactly compensates the divergent terms of S_1^{SM}, so that the contribution S_1^{TD} of the quantum field φ to the thermodynamical entropy of a black hole is of order of $O(\varepsilon^0)$. In particular, it means that S_1^{TD} is independent of the nature of the cutoff ε, which is assumed in S_1^{SM} and which for its calculation requires knowledge of physics at the Planckian scale. In other words, the thermodynamical entropy of a black hole is completely determined by low energy physics. S_1^{TD} contains the part which depends on r_B. This part describes the entropy of thermal gas of quanta of φ field, located outside a black hole within the cavity of size r_B. In addition, S_1^{TD} also contains part independent of r_B describing quantum corrections to the black hole entropy. For black holes of mass much larger than the Planckian mass these corrections are much smaller than A/l_P^2 and can be neglected. As the result of the above described compensation mechanism, the dynamical degrees of freedom of the black hole practically do not contribute to its thermodynamical entropy S^{TD}, and the latter is defined by the (renormalized) tree-level quantity S^{BH}.

To make the basic idea clearer we restricted ourselves, in the above discussion, by considering a nonrotating black hole. The analysis is easily applied to the case of a charged rotating black hole as well as to their non-Einsteinian and n-dimensional generalizations.

To summarize, it has been shown that the Bekenstein-Hawking entropy does not coincide with the statistical-mechanical entropy $S_1^{\mathrm{SM}} = -\mathrm{Tr}(\hat{\rho}\ln\rho)$ of a black hole. The latter entropy is determined by the internal degrees of freedom of the black hole, describing different states which may exist inside a black hole for the same value of its external parameters. The discrepancy arises because in the state of thermal equilibrium the parameters of internal degrees of freedom of a black hole depend on the temperature of the system in the universal way. This results in the universal cancellation of all those contributions to the thermodynamical entropy which depend on the particular properties and number of fields. That is why the thermodynamical entropy of black holes in Einstein's theory is always S^{BH}.

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Note added.—After the present paper was submitted to publication, the paper by D.V. Fursaev appeared as Report No. DSF-32/94 (hep-th/9408066). In this paper it was argued that for $\beta \neq \beta_H$ δ-like curvature connected with a cone singularity gives an additional temperature dependent contribution to the free energy. If one uses such a modified free energy to define entropy one would get $\beta^2 \partial F_1/\partial\beta_{r_+} = S_1^{\mathrm{SM}} + s$, where s is connected with cone contribution. As the result ΔS_1 in Eq. (11) of the present paper would contain explicit dependence on β. Nevertheless, the conclusion of the present paper that $S^{\mathrm{TD}} \neq S^{\mathrm{SM}}$ remains unchanged. For $\beta = \beta_H$, the cone singularity disappears and the extra terms in the free energy discussed by Fursaev vanish. So that the mechanism of compensation of the divergent near the horizon one-loop contribution to S^{TD} discussed in the present paper also remains valid.

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