Probability of an Eigenvalue Number Fluctuation in an Interval of a Random Matrix Spectrum

M. M. Fogler and B. I. Shklovskii
Theoretical Physics Institute, University of Minnesota, 116 Church Street Southeast, Minneapolis, Minnesota 55455
(Received 12 October 1994)

We calculate the probability to find exactly n eigenvalues in a spectral interval of a large random $N \times N$ matrix when this interval contains $s \ll N$ eigenvalues on average. The calculations exploit an analogy to the problem of finding a two-dimensional charge distribution on the interface of a semiconductor heterostructure under the influence of a split gate.

PACS numbers: 05.40.+j, 02.50.–r

Random-matrix theory found its applications in numerous branches of physics. Among them are the statistical theory of slow neutron resonances, the theory of chaotic systems, and the properties of small metallic samples (mesoscopic physics), etc. [1].

The random-matrix theory studies statistical properties of spectra of large matrices whose elements have a given random distribution. Much interest has been focused on the properties of a spectral interval containing on average s eigenvalues, where s is a small fraction of total number of eigenvalues N. The actual number n of eigenvalues in this interval fluctuates from one realization of a random matrix to another. Among different statistical characteristics of these fluctuations, two, in particular, were the objects of intensive study. One of them was the variance of n: $\langle \delta n^2 \rangle = \langle (n - s)^2 \rangle$. Using a thermodynamical argument Dyson [2] showed that

$$\langle \delta n^2 \rangle = \frac{2}{\pi^2 \beta} \ln s + B_\beta \,, \tag{1}$$

where $\beta = 1, 2$, or 4 for the three possible ensembles of random Hamiltonians: orthogonal, unitary, and symplectic, respectively. The constant B_β was calculated by Dyson and Mehta [3]. It is different for different ensembles, but is generally of order unity. Here we quoted Dyson's result in its simplest form when possible degeneracies and series of noninteracting levels are neglected. The fact that $\langle \delta n^2 \rangle \ll s$ is the consequence of a so-called level repulsion.

Another popular quantity was the probability density $P(d)$ to find two consecutive levels separated by a distance of d average level spacings. For the orthogonal ensemble $P(d)$ is well approximated by the famous "Wigner surmise" $P(d) = (\pi d/2)\exp(-\pi d^2/4)$ [4], if d is not too large. The level repulsion can be seen in the fact that $P(d) \to 0$ as $d \to 0$. For large d the asymptotic behavior of $P(d)$ is different from that of Wigner surmise [1,3]

$$\ln P(d) = -\frac{\pi^2}{16} d^2 - \frac{\pi}{4} d + O(\ln d) \,. \tag{2}$$

In this paper we calculate the asymptotical behavior of a more informative quantity $E_\beta(n,s)$ [1], which is the probability of finding exactly n eigenvalues in a spectral interval containing s eigenvalues on average. It is easy to see that $P(s) = d^2 E_\beta(0,s)/ds^2$ and $\langle \delta n^2 \rangle = \sum_{n=0}^{\infty} (n - s)^2 E_\beta(n,s)$. Explicit formulas for $E_\beta(n,s)$ exist only for the case $n \ll s$ [5,6]. Here we derive $E_\beta(n,s)$ for an arbitrary relation between n and s using the method of an electrostatic analogy [2]. The limitation of this method is that it is valid for $n,s \gg 1$.

As an important result we present the expression for $E_\beta(n,s)$ in the limit $|n - s| \ll s$:

$$\ln E_\beta(n,s) = -\frac{\pi^2 \beta}{4} \frac{\delta n^2}{\ln(8s/|\delta n|) + B(\delta n)} \,, \tag{3}$$

where $\delta n = n - s$ and B weakly depends on δn. This distribution is nearly Gaussian for $\delta n^2 \ll \ln s$, and Eq. (1) follows. The Gaussian form agrees with the conjecture put forward by Altshuler *et al.* [7]. For $\delta n^2 \gg \ln s$ the probability of a fluctuation δn in the eigenvalue number is, however, significantly smaller than as it would be determined from a Gaussian distribution with the variance given by Eq (1).

Let us now turn to the derivation of the general expression for $E_\beta(n,s)$. As discussed by Dyson [2], the joint probability density function of the eigenvalues x_i of a random matrix can be written in the form

$$Q(x_1, \ldots, x_N) = C \exp(-\beta W) \,, \tag{4}$$

where

$$W = \sum_i \phi_c(x_i) - \sum_{i<j} \ln |x_i - x_j| \,, \tag{5}$$

and C is a normalizing constant. This form is nothing else as the thermodynamic Gibbs factor of the system with Hamiltonian W at temperature $1/\beta$. This system is a gas of N point charges. Each pair of charges i and j exhibits the electrostatic repulsion logarithmically dependent on the distance $|x_i - x_j|$ between them. The confinement of the gas is provided by an external potential $\phi_c(x)$. Once this analogy is established, one can apply the methods of the classical thermodynamics for finding various statistical

 0031-9007/95/74(17)/3312(4)$06.00

properties of random matrix spectra. The routine is to introduce the partition function

$$\psi_\beta = \frac{1}{N!} \int \cdots \int e^{-\beta W} dx_1 \cdots dx_N . \quad (6)$$

The factor $1/N!$ corresponds to our treatment of charges as indistinguishable particles [2]. The quantity $E_\beta(n,s)$ is the ratio of two values of ψ_β calculated for two different domains of the integration. The first domain is determined by the condition that exactly n charges belong to the given interval of length s. The second value corresponds to the unrestricted domain of integration.

When both n and s are large one attempts to abandon the discreet formulation of the problem in favor of the continuum one. Following Dyson we make three assumptions: (i) There is a macroscopic eigenvalue density function $\rho(x)$. (ii) For a given $\rho(x)$ the free energy of the gas is composed of two parts

$$F = V_1 + V_2, \quad (7)$$

where V_1 is the potential energy in the mean-field approximation

$$V_1 = \int \int \rho(x) \left[\frac{1}{2} \phi(x) + \phi_c(x) \right] dx . \quad (8)$$

The quantity $\phi(x)$, defined as

$$\phi(x) = - \int \rho(y) \ln|x - y| dy , \quad (9)$$

has the meaning of the electrostatic potential. The other contribution V_2 depends on the local density

$$V_2 = \int f[\rho(x)]\, dx , \quad (10)$$

$f[\rho]$ being the free energy per unit length of a Coulomb gas with uniform density ρ. As Dyson showed [2] (see also Mehta's book [1])

$$f[\rho] = (1/\beta - 1/2)\rho \ln\rho . \quad (11)$$

The first term is the entropy multiplied by the temperature and the second term is the correlation energy per unit length (the difference between the energies of the Wigner crystal and the continuous charge distribution with the same average density for the logarithmic law of interaction). (iii) The overwhelmingly dominant contribution to the integral (6) comes from configurations not deviating significantly from an "optimal" density fluctuation that makes F a minimum subject to

$$\int_{-t}^{t} \rho(x)\, dx = n , \quad (12)$$

where $t = s/2$.

Another simplification may be obtained by noting that if $s \ll N$ then we study only a small fraction of the whole spectral interval. In this case the explicit form of the confinement potential in the expression (5) is irrelevant and can be chosen according to our needs. We find it the most convenient to replace the confinement potential ϕ_c by a compensating background of unit density. The appropriate modification of Eq. (8) is then

$$V_1 = -\frac{1}{2} \int \int [\rho(x) - 1][\rho(y) - 1] \ln|x - y| dx\, dy . \quad (13)$$

The probability $E_\beta(n,s)$ is expressed as

$$\ln E(n,s) \approx -\beta \min F , \quad (14)$$

where F is given by Eqs. (7), (10), (13), and (11) and the minimum is sought in the class of continuous nonnegative functions $\rho(x)$ under the condition (12). This optimal fluctuation is very close to the one which ensures the lowest value of the mean-field part V_1 of the free energy F. However, the optimization has to be done taking into account the local part V_2 as well, and we will make an error of order $O(\ln|\delta n|)$ as long as $s \gg 1$ and $|\delta n| \gg 1$. As discussed by Dyson [2], we lost such order terms anyway upon the transition from the discrete to the continuum formulation of the problem.

Using the variational principle we find that the electrostatic potential $\phi(x)$ created by the optimal density fluctuation must satisfy the following conditions:

$$\begin{aligned} \phi(x) &= -V, \quad |x| < t,\ \rho(x) > 0 , \\ &> -V, \quad |x| < t,\ \rho(x) = 0 , \\ &= 0, \qquad |x| > t , \end{aligned} \quad (15)$$

if $n < s$ and, similarly,

$$\begin{aligned} \phi(x) &= V, \quad |x| < t,\ \rho(x) > 0 , \\ &< V, \quad |x| > t,\ \rho(x) = 0 , \\ &= 0, \qquad |x| > t , \end{aligned} \quad (16)$$

otherwise. The meaning of these conditions is that the Coulomb gas breaks into "metallic" regions, where the potential is perfectly screened, and "insulating" regions, where the charge density $\rho(x)$ vanishes and there is no screening. The quantity V is some constant [it is a Lagrange multiplier corresponding to the requirement (12)].

Consider the case $n < s$ first. We notice that this problem is equivalent to another one, which at first glance appears completely different. We are referring to the problem of finding the charge distribution of a laterally confined two-dimensional electron gas (2DEG) in a heterostructure. The reason for this is as follows. Because of a large aspect ratio in some such devices, they may be considered as translationally invariant in one of the dimensions. Then the usual Coulomb interaction of two-dimensional charges leads to the logarithmic interaction in terms of the one-dimensional charge density. This is precisely the type of interaction that we are having in our Coulomb gas model.

Consider a simplified model of the split-gate device on the GaAs/Al_xGa_{1-x}As heterostructure studied in Ref. [8]. In this model (see Fig. 1), ionized donors and the two-dimensional electron gas are characterized by continuous charged densities. The donors constitute a uniform positive background compensated by the 2DEG. The split gate is represented by two semi-infinite metal planes separated by the gap of width $2t$ centered at $x = 0$. The system is translationally invariant in $\hat{y}$ direction and all the charges and the gate are in the plane $z = 0$.

Suppose a voltage difference V is applied between the gate and the 2DEG. The 2DEG under the gate is depleted and confined to a strip $-b < x < b$. In the regions $b < |x| < t$ the charge density is due to the background only. If we subtract the total charge density from the background charge density, then this new quantity will be nonzero in three disconnected regions: in a central strip of 2DEG and in the metallic gate. Considering the analogy discussed above, we designate this quantity by the *same symbol* $\rho(x)$ as the eigenvalue density of a random matrix. Let the width of the strip of the 2DEG be $2b$, then the $\rho(x)$ in units of the background charge density is [8]

$$\rho(x) = \mathrm{Re}\sqrt{(b^2 - x^2)/(t^2 - x^2)}. \tag{17}$$

The consequence for the distribution of the eigenvalue density in the optimal fluctuation is now that it is clearly given by the same formula (17). The parameter $b < t$ can be found from Eq. (12). The graph of $\rho(x)$ is shown in Fig. 2(a). Note that for $n = s$, $b = 0$, and $\rho(x) = 0$ in the interval $-t < x < t$. This particular case was studied by Dyson [2] to obtain the asymptotic form of $E_\beta(0, s)$ and $P(s)$.

To calculate the free energy corresponding to the fluctuation we need to know the distribution of the electric field in the Coulomb gas model. It can also be obtained by knowing the corresponding distribution for the split-gate device. Namely, the field in the Coulomb gas is by the factor of 2 weaker. The reason for this is that the potential due to real two-dimensional charges is $-2\int \rho(y) \ln|x - y|$, whereas the potential (9) in the Coulomb gas model is twice as small. With this correction the electric field in the Coulomb gas model is

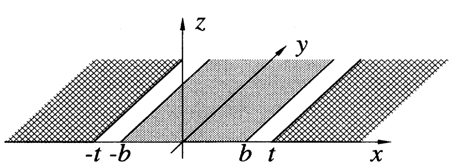

FIG. 1. The model of the split-gate device. The two shaded areas represent the gate in the shape of two semi-infinite planes. Beneath the gate the two-dimensional electron gas located in $z = 0$ plane is depleted. The density of the electron gas does not vanish only in a narrow strip (grey area) between the two halves of the gate.

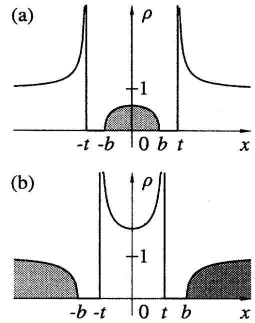

FIG. 2. The optimal eigenvalue density fluctuation $\rho(x)$ for (a) $n < s$, (b) $n > s$. The grey and blank regions show the locations of the 2DEG charge and the gate charge, respectively, in the equivalent gated devices.

given by the expression

$$-d\phi(x)/dx = \pi\ \mathrm{Im}\sqrt{(b^2 - x^2)/(t^2 - x^2)}. \tag{18}$$

The electric field vanishes everywhere where $\rho(x)$ is nonzero, in agreement with Eq. (15).

The case $n > s$ can be reduced to a similar electrostatic problem. The gate depleting the 2DEG now has the geometry of an infinite strip of width $2t$. The 2DEG survives only at $|x| > b$ with $b > t$, and in the regions $t < |x| < b$ $\rho(x)$ is zero. Returning back to the Coulomb gas model, we find that the eigenvalue density [Fig. 2(b)] and the electric field of the optimal fluctuation are given by the same expressions (17) and (18) as in the case $n < s$.

The calculation of the parameters of the optimal fluctuations is straightforward and the answer can be given in terms of the complete elliptic integrals [9]

$$V_1 = \frac{\pi^2 t^2}{4} - \frac{Vn}{2}, \quad V = \pi t[E(k) - k'^2 K(k)],$$
$$n = 2t[E(k') - k^2 K(k')], \tag{19}$$

when $n < s$ and

$$V_1 = -\frac{\pi^2 t^2}{4}\frac{k^2}{k'^2} + \frac{Vn}{2}, \quad V = \frac{\pi t}{k'}[K(k) - E(k)],$$
$$n = \frac{2t}{k'}E(k'), \tag{20}$$

when $n > s$. Here, $k^2 = 1 - k'^2$ and k' is defined as $k' = b/t$ for $n < s$ and $k' = t/b$ for $n > s$. Also $V_2 = V$ in both cases. Equation (14) now reads

$$\ln E_\beta(n,s) = -\beta V_1 - \left(1 - \frac{\beta}{2}\right)V + O(\ln|n - s|). \tag{21}$$

To use this formula one first has to find the k corresponding to given n and s and then substitute this value into the expressions for V and V_1. For example, for $n \ll s$ one finds that $k' \approx (2n/\pi t)^{1/2}$ and then $V \approx \pi t - (n/2)[\ln(8\pi t/n) + 1]$. As a result,

$$\ln E_\beta(n,s) \approx -\frac{\beta}{4}\pi^2 t^2 + \left(\beta n - 1 + \frac{\beta}{2}\right)\pi t - \frac{\beta}{8}n^2$$
$$+ \frac{n}{2}\left(\frac{\beta}{2}n - 1 + \frac{\beta}{2}\right)\left(\ln\frac{n}{8\pi t} - 1\right). \tag{22}$$

This formula coincides with results given in Refs. [5,6] and reproduces Eq. (2) upon the substitution $d = 2t$ and $n = 0$.

Consider now the case of small fluctuations $|\delta n| \ll s$, which corresponds to $k \ll 1$. In this limit the expansion of Eqs. (19) and (20) in asymptotic series yields

$$V_1 \approx \frac{\pi^2}{8}(tk^2)^2\left(\ln\frac{4}{k} + \frac{1}{4}\right), \quad V \approx \frac{\pi}{2}tk^2,$$
$$|\delta n| \approx tk^2\left(\ln\frac{4}{k} + \frac{1}{2}\right).$$

Using Eq. (21) one recovers Eq. (3). As a possible method of the numerical check of the latter, we suggest calculating the ratio of two subsequent even moments M_{2m+2} and M_{2m} of $E_\beta(n,s)$ defined as $M_{2m} = \sum_{n=0}^{\infty}(n - s)^{2m}E_\beta(n,s)$. This ratio is expected to be

$$\frac{M_{2m+2}}{(2m+1)M_2M_{2m}} \simeq 1 - \frac{1}{2}\frac{\ln 2am}{\ln s}, \tag{24}$$

where $a = 2/\pi^2\beta$ and $1 \ll am \ll s^2$. Whereas for the Gaussian distribution this ratio is just equal to unity, in our case it decreases with m. For instance, at $m \sim s/2a$ it should be about 0.5.

For completeness we provide also the asymptotic form of $\ln E_\beta(n,s)$ for *large* positive δn, i.e., for $n \gg s$:

$$\ln E_\beta(n,s) = -\frac{\beta n^2}{2}\left(\ln\frac{8n}{\pi s} - \frac{3}{2}\right) - \left(1 - \frac{\beta}{2}\right)n\left(\ln\frac{8n}{\pi s} - 1\right) + O(\ln n). \tag{25}$$

One observes that the main terms in both Eqs. (3) and (25) are quadratic in δn and contain large logarithmic factors. This can be explained in terms of the electrostatic analogy as follows. The quantity $\ln E_\beta(n,s)$ is up to a constant of the work required to charge a two-dimensional capacitor by δn charge units. This work is equal to $\delta n^2/2C$, where C is the capacitance. In the case of $|\delta n| \ll s$ the width s of the central plate of this capacitor is much larger than the gap between the plates and C is logarithmically large. In the case of $n \gg s$ the situation with geometrical parameters of the capacitor is reversed, and its capacitance is thus inversely proportional to a large logarithmic factor.

In conclusion, we studied the probability $E_\beta(n,s)$ to find a given number n of eigenvalues in an interval of a random matrix spectrum defined by the condition that this interval contains s eigenvalues on average. We calculated the asymptotical behavior of $E_\beta(n,s)$ for large s. It is found to be Gaussian for small fluctuations of n around its average value and to decay faster than Gaussian at $\delta n^2 \gtrsim \ln s$. We suggested a method of the numerical verification of our result based on calculating of large order moments of $E_\beta(n,s)$.

This work was supported by NSF under Grant No. DMR-9321417.

Note added.—After the submission of this paper, the authors received a preprint from Professor Dyson [10] where in Section II he obtained the identical results (without using the analogy to the semiconductor devices).

[1] See M.L. Mehta, *Random Matrices* (Academic Press, Boston, 1990), 2nd ed., and references therein.
[2] F.J. Dyson, J. Math. Phys. **3**, 140 (1962); **3**, 157 (1962); **3**, 166 (1962).
[3] F.J. Dyson and M.L. Mehta, J. Math. Phys. **3**, 701 (1962).
[4] E.P. Wigner, in Proceedings of the Gatlinberg Conference on Neutron Physics by Time of Flight, Gatlinberg, 1956 (Oak Ridge National Laboratory Report No. ORNL-2309), p. 67.
[5] E.L. Basor, C.A. Tracy, and H. Widom, Phys. Rev. Lett. **69**, 5 (1992); C.A. Tracy and H. Widom, in *Geometric and Quantum Aspects of Integrable Systems*, edited by G.F. Helmnick (Springer-Verlag, Berlin, 1993), pp. 103-130.
[6] G. Mahoux and M.L. Mehta, J. Phys. I (France) **3**, 697 (1993).
[7] B.L. Altshuler, I.Kh. Zharekeshev, S.A. Kotochigova, and B.I. Shklovskii, Zh. Eksp. Teor. Fiz. **94**, 343 (1988)[Sov. Phys. JETP **67**, 625 (1988)].
[8] I.A. Larkin and V.B. Shikin, Phys. Lett. A **151**, 335 (1990).
[9] P.F. Byrd and M.D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists* (Springer-Verlag, Berlin, 1971).
[10] F.J. Dyson, Report No. IASSNSS-HEP-92/43, 1992, (unpublished).