# Can Sigma Models Describe Finite Temperature Chiral Transitions?

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Large-$N$ expansions and computer simulations indicate that the universality class of the finite-temperature chiral symmetry restoration transition in the 3D Gross-Neveu model is mean-field theory. This is a counterexample to the standard "sigma model" scenario which predicts the 2D Ising model universality class. We trace the breakdown of the standard scenario (dimensional reduction and universality) to the absence of canonical scalar fields in the model. We point out that our results could be generic for theories with dynamical symmetry breaking, such as quantum chromodynamics.

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When studying the finite-temperature chiral restoration transition in QCD one is usually guided by the concepts of dimensional reduction and universality. A compelling idea, first put forward in Ref. [1] and later elaborated on in Ref. [2], is that in four-dimensional QCD with $N_f$ light quarks the physics near the chiral transition can be described by the three-dimensional $\sigma$ model with the same global symmetry. The reasoning behind this proposal is based on counting the light degrees of freedom and can be phrased as follows. The transition region is dominated by the longitudinal and transverse fluctuations of the order parameter, $\sigma$ and $\pi$, which go soft at the transition temperature. Being bosonic, $\sigma$ and $\pi$ have zero modes, $\omega_n = 0$, in their finite-temperature Matsubara decomposition. These zero modes are the only relevant degrees of freedom in the scaling region and at low energies the $n \neq 0$ modes decouple. Therefore, in the context of a $d$-dimensional theory, one concludes that the phase transition is described by an effective scalar theory in $d - 1$ dimensions. As a consequence, the chiral transition of four-dimensional QCD, with $N_f = 2$ flavors, should lie in the same universality class as a three-dimensional O(4) magnet [1,2]. Similarly, other models, e.g., four-fermion theories in $d$ dimensions such as the Gross-Neveu model [3] with discrete symmetries or the Nambu–Jona-Lasinio model [4] with continuous chiral symmetries, are expected to be in the universality class of a $(d - 1)$-dimensional Ising or Heisenberg magnet, respectively.

It is the purpose of this Letter to discuss the assumptions underlying this analysis. As an illustration we study two examples: a purely bosonic theory, an O($N$) $\sigma$ model where the ideas of dimensional reduction apply, and a Gross-Neveu model with composite scalars where they fail. We discuss the generic features of the models that might apply to other field theories at finite temperature. At the end we comment on the implications these two examples have on QCD.

To illustrate how the idea of dimensional reduction is realized in scalar theories, we start with the $N$-component scalar theory and consider the large-$N$ limit [5] for simplicity. To avoid complications due to Goldstone bosons, we work in the symmetric phase. At zero temperature, to leading order in $1/N$, the corrections to the propagator are given by the single tadpole contribution. The susceptibility, $\chi = \int_x \langle \phi(x)\phi(0)\rangle_c$, is the zero-momentum projection of the correlation function. Since the tadpole is momentum independent, it affects only the susceptibility and not the wave function renormalization. Defining the critical curvature $\mu_c^2$ as the point where the susceptibility diverges ($\mu_c^2 + \lambda \int_q 1/q^2 = 0$), the expression for the inverse susceptibility can be recast into

$$\chi^{-1}\left(1 + \lambda \int_q \frac{1}{q^2(q^2 + \chi^{-1})}\right) = \mu^2 - \mu_c^2, \qquad (1)$$

where $\int_q = \int d^dq/(2\pi)^d$, and we absorb the combinatorial factor in $\lambda$. The extraction of the critical index $\gamma$ reduces to counting powers of the infrared (IR) singularities on the left-hand side of Eq. (1). Above four dimensions, both terms are IR finite and the scaling is mean field ($\gamma = 1$). Below four dimensions the second term in Eq. (1) dominates the scaling region—the integral diverges as $\chi^{(4-d)/2}$. This gives the zero-temperature susceptibility exponent $\gamma = 2/(d - 2)$ [5].

At finite temperature, apart from the replacement of the frequency integral with the Matsubara sum, modifications are minimal [6]. For a given value of $\mu^2$ we define the critical temperature $T_c$ by $\mu^2 + \lambda T_c \sum_n \int_{\mathbf{q}} 1/(\omega_{nc}^2 + \mathbf{q}^2) = 0$, where $\omega_{nc} = 2\pi n T_c$. The momentum integrals are now performed over $d - 1$ dimensional space. Separating the $n = 0$ mode ($\omega_0 = 0$) from the rest of the sum, we get the leading singular behavior

$$\chi^{-1}\left(1 + \lambda T_c \int_{\mathbf{q}} \frac{1}{\mathbf{q}^2(\mathbf{q}^2 + \chi^{-1})} + \sum_{n\neq 0}\cdots\right) = \lambda T_c \int_{\mathbf{q}} \frac{T/T_c - 1}{\mathbf{q}^2 + \chi^{-1}} + \sum_{n\neq 0}\cdots. \qquad (2)$$

The $n = 0$ piece dominates the scaling region. It resembles the zero-temperature expression, Eq. (1), except that now, the integrals are performed in $d - 1$ dimensions, instead of $d$. The power counting is the same as before and it yields the thermal exponent $\gamma_T = 2/(d - 3)$ which is the same as the zero temperature $\gamma$ in $d - 1$ dimensions [6]. It is easy to obtain the other critical exponents; they show the same type of behavior as $\gamma$.

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To illustrate how compositeness affects the physics near the phase transition, we analyze the problem of chiral symmetry restoration in a Gross-Neveu model given by the Lagrangian $L = \overline{\psi}(i\partial\!\!\!/ + m + g\sigma)\psi - \frac{1}{2}\sigma^2$, where notation is the Euclidean transcription of the standard one [3]. Besides being an interesting theoretical model, it is also believed that, when properly extended to incorporate continuous chiral symmetry, four-fermion models are more realistic as effective theories of QCD than the linear sigma model, especially at scales where quark substructure is important. When fermions are integrated out of the Gross-Neveu model, the Ising symmetry $\sigma \to -\sigma$ of the effective action becomes manifest. If the dimensional reduction + universality arguments hold [1], the finite-temperature transition of the $d$-dimensional model would lie in the universality class of the $(d-1)$-dimensional Ising model. In the remainder of the Letter we explain how and why this argument fails.

First, we start with the zero-temperature gap equation and corresponding critical exponents. The model can be treated in the large-$N$ limit. To leading order, the fermion self-energy $\Sigma$ comes from the $\sigma$ tadpole: $\Sigma = m - g^2\langle\overline{\psi}\psi\rangle$. To obtain the scaling properties of the theory, we define the critical coupling as $1 = 4g_c^2 \int_q 1/q^2$. Combining this definition with the gap equation leads to

$$\frac{m}{\Sigma} + (g^2/g_c^2 - 1) = 4g^2 \int_q \frac{\Sigma^2}{q^2(q^2+\Sigma^2)}. \tag{3}$$

Like the scalar example, this form is especially well suited for extracting critical indices since the problem reduces again to the counting of the infrared divergences on the right-hand side [7,8]. The critical indices are defined by $\langle\overline{\psi}\psi\rangle|_{m=0} \sim t^\beta, \langle\overline{\psi}\psi\rangle|_{t=0} \sim m^{1/\delta}, \Sigma|_{m=0} \sim t^\nu$, etc. Here, $t = g^2/g_c^2 - 1$ is the deviation from the critical coupling. Since $\Sigma \sim \langle\overline{\psi}\psi\rangle, \beta = \nu$ to leading order. Above four dimensions the integral in Eq. (3) is finite in the limit of vanishing $\Sigma$ and the scaling is mean field.

Below four dimensions, the $\Sigma \to 0$ limit is singular—the integral scales as $\Sigma^{d-2}$. Thus, in the chiral limit $t \sim \Sigma^{d-2}$, and at the critical point $t = 0$ away from the chiral limit, $m \sim \Sigma^{d-1}$. The resulting exponents are non-Gaussian: $\beta = 1/(d-2)$ and $\delta = d-1$. The remaining exponents are obtained easily: $\eta = 4-d, \gamma = 1$ [7,8] and one can check that they obey hyperscaling.

We now consider the Gross-Neveu model at finite temperature. We choose to stay between two and four dimensions to emphasize how zero-temperature power-law scaling changes at finite temperature. The gap equation is now modified to

$$\Sigma = m + 4Tg^2 \sum_n \int_{\mathbf{q}} \frac{\Sigma}{\omega_n^2 + \mathbf{q}^2 + \Sigma^2}, \tag{4}$$

where $\omega_n = (2n+1)\pi T$. For $g > g_c$ the critical temperature is determined by $1 = 4T_c g^2 \sum_n \int_{\mathbf{q}} 1/(\omega_{nc}^2 + \mathbf{q}^2)$, where $\omega_{nc} = (2n+1)\pi T_c$. This expression defines a critical line in the $(g,T)$ plane. For every coupling there exists a critical temperature beyond which the symmetry is restored. Conversely, for a fixed temperature there is a critical coupling, defined by the above expression, corresponding to symmetry restoration. At zero temperature, the symmetry is restored at $g = g_c$. Thus $(g = g_c, T = 0)$ is the ultraviolet (UV) fixed point. As the coupling moves away from $g_c$, a higher restoration temperature results. At infinite coupling the end point $(g = \infty, T = T_c)$ is the IR fixed point. The critical line connects the UV and IR fixed points dividing the $(g,T)$ plane into two parts. [The equation for the critical line can be brought into a compact form by combining the expression for $T_c$ with the definition of the zero-temperature critical coupling. This results in $(g^2/g_c^2 - 1) \sim T_c^{d-2}(g)$, i.e., $T_c(g) \sim \Sigma(T=0)$. In this way, for any value of the coupling, the critical temperature remains the same in physical units.]

Combining the definition of $T_c$ with the finite-temperature gap equation, we can bring it to a form similar to Eq. (3):

$$\frac{m}{\Sigma} = (1 - T/T_c) + 4Tg^2 \sum_n \int_{\mathbf{q}} \frac{\Sigma^2 + \omega_{nc}(\omega_n + \omega_{nc})(T/T_c - 1)}{(\omega_{nc}^2 + \mathbf{q}^2)(\omega_n^2 + \mathbf{q}^2 + \Sigma^2)}. \tag{5}$$

The extraction of the critical exponents proceeds along the same lines as in the zero-temperature case. One difference relative to Eq. (3) becomes apparent immediately: The zero modes are absent here and the integrand in Eq. (5) is regular in the $\Sigma \to 0$ limit even below four dimensions. Consequently, the IR divergences are absent from all the integrals and the scaling properties are those of mean-field theory: $\beta = \nu = \frac{1}{2}, \delta = 3$, etc. This is true for any $d$ below or above four. It appears that in this case, contrary to the scalar example, the effect of making the temporal direction finite $(1/T)$ is to regulate the IR behavior and suppress fluctuations. This is manifest in other thermodynamic quantities as well. For example, to leading order, the scalar susceptibility $\chi = \partial\langle\overline{\psi}\psi\rangle/\partial m$ is given by

$$\chi^{-1} = 8g^2 T \sum_n \int_{\mathbf{q}} \frac{\Sigma^2}{(\omega_n^2 + \mathbf{q}^2 + \Sigma^2)^2}. \tag{6}$$

Once again, because of the absence of the zero mode $(\omega_0 = \pi T)$, the integral in Eq. (6) is analytic in $\Sigma$, and the mean-field relation $\chi^{-1} \sim \Sigma^2$ follows. This is equivalent to $\gamma = 2\nu = 1$. The explicit calculation of the momentum dependence of the $\sigma$ propagator [9] yields $\eta = 0$.

The scaling laws of the finite-temperature transition obtained above are completely different from the predictions of Ref. [1]. In fact, even the systematics are opposite. The regulating character of the temperature drives the lower dimensional theory towards an effective theory that has Gaussian critical exponents.

The fermionic model discussed above was first analyzed in Ref. [9]. Higher order calculations have shown that the results are not artifacts of the large-$N$ limit [10]. In addition, it was explained in Ref. [10] how the Ising point is recovered in four-dimensional Yukawa models beyond the leading order in $1/N$ and why this does not happen in Gross-Neveu models. Lattice simulations of the three-dimensional model have verified the predictions of the large-$N$ expansion at zero temperature, at nonzero temperature, and at nonzero chemical potential [11]. The results for critical indices have been verified and improved by larger scale simulations enhanced by histogram methods [12]. We have done additional simulations to check the finite-temperature results [11] in detail. Lattices of sizes $6 \times 30^2, 6 \times 72^2, 12 \times 36^2$, and $12 \times 72^2$ were simulated at $N = 12$ using the hybrid Monte Carlo algorithm described in Ref. [11]. High statistics runs (several tens of thousands of trajectories for each coupling) were made on a variety of lattices to guarantee that the simulations were probing the physical IR modes at finite temperature. Luckily, our task is to distinguish mean-field exponents from those of the two-dimensional Ising model, and, as reviewed in Table I, they are dramatically different. We will discuss the exponents $\delta$ and $\beta$ (defined above) and leave other calculations to a lengthier presentation. The order parameter $\sigma$ was measured over a range of couplings and fits of the form $\sigma = A(1/g^2 - 1/g_c^2)^\beta$ were made to determine critical points and indices. In Fig. 1 we show $d\ln\sigma^2$, the difference of the logarithm of $\sigma^2$, plotted against $d\ln t$, the difference of the reduced coupling $t = |1/g^2 - 1/g_c^2|/1/g_c^2$ for the $6 \times 72^2$ and $12 \times 72^2$ lattices. Vacuum tunneling ($\sigma$ flipping to $-\sigma$) occurs when $1/g^2$ is chosen too close to $1/g_c^2$ on a finite lattice and this effect restricts the range of $t$ shown in the figure. The data are in excellent agreement with mean-field theory where $\beta = \frac{1}{2}$, and rule out the Ising model value of $\frac{1}{8}$. Note that the statistical error bars in the figure are smaller than the plotting symbols themselves. Since all lattice sizes give the same estimates of $\beta$ while their critical temperatures are quite different, we are confident that the simulation is probing the true continuum behavior of the finite-temperature transition and is not corrupted by a sluggish crossover between symmetric and asymmetric lattices. Next, we measured the response of the order parameter at criticality to an external symmetry breaking field (bare fermion mass) and obtained $\delta$. The data are shown in Fig. 2 for lattice sizes $6 \times 30^2$ and $12 \times 36^2$. We found $\delta = 3.1(1)$. The Ising model value of $\delta = 15$ is ruled out. In all of these calculations we carefully visualized the $\sigma$ field to check for nonuniform configurations that would violate the mean-field hypothesis [13]. None were found and all the past simulations [11] and the new ones reported here support the contention that the large-$N$ results are reliable for this problem.

TABLE I. Critical exponents of the 3D Gross-Neveu and 2D Ising model.

| | $d = 3$ $T = 0$ | Gross, Neveu, $T \neq 0$ | $d = 2$ Ising |
|---|---|---|---|
| $\beta$ | 1 | $\frac{1}{2}$ | $\frac{1}{8}$ |
| $\delta$ | 2 | 3 | 15 |
| $\gamma$ | 1 | 1 | $\frac{7}{4}$ |
| $\nu$ | 1 | $\frac{1}{2}$ | 1 |
| $\eta$ | 1 | 0 | $\frac{1}{4}$ |

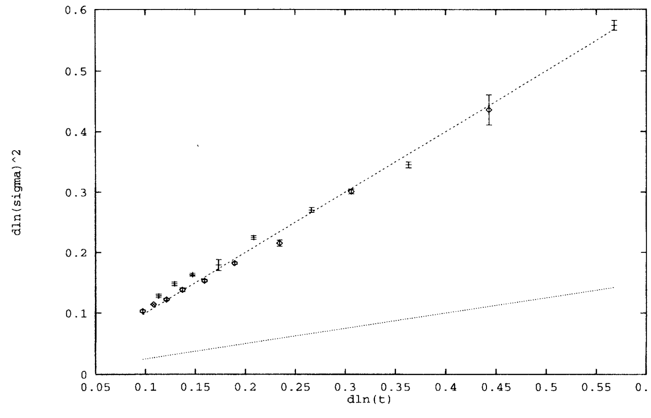


FIG. 1. $d\ln\sigma^2$ vs $d\ln t$ on $6 \times 72^2$ (diamonds) and $12 \times 72^2$ (crosses) lattices. The dotted line is the Ising model.

An important feature of the exponents corresponding to the finite-temperature transition is that they violate hyperscaling [14]. Usually, hyperscaling violation occurs above four dimensions and is expressed in terms of exponent inequalities [14], e.g., $2\beta\delta - \gamma \leq d\nu$. Strict inequality is applicable only for $d > 4$ and implies factorization of the correlation functions. In our example the above inequality goes in the opposite direction and the breakdown of hyperscaling is not accompanied by the factorization of Green's functions.

One might be tempted to draw an analogy between the present problem and the superconducting transition in BCS systems where the scaling region is very narrow and, until recently, the observed scaling was believed to be mean field. There is, however, an essential difference between the two cases. In superconductors, the origin of a narrow critical region is the fact that the range of the interaction scales as the size of Cooper pairs which dissociate at the transition point. Close to the transition the size of the Cooper pairs increases, and the range of interaction becomes progressively longer causing the shrinking of the scaling region (see, e.g., Ref. [15]).

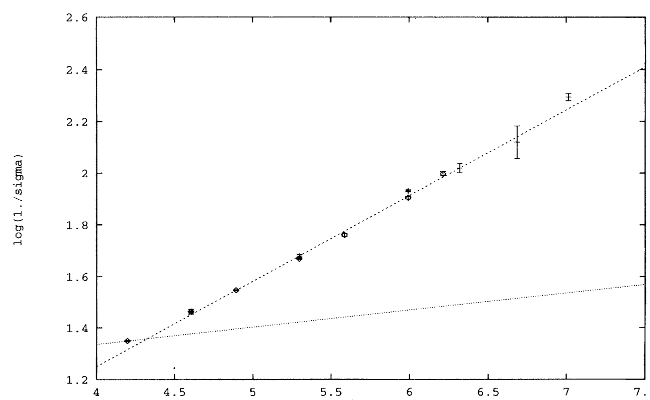


FIG. 2. Order parameter response at criticality plotted against bare fermion mass on $6 \times 30^2$ (diamonds) and $12 \times 36^2$ (crosses) lattices. The dotted line is the Ising model.

In the zero-temperature Gross-Neveu model, on the other hand, massless fermions generate long-range forces. This is manifested through the appearance of noninteger powers of the gradient of the $\sigma$ field in the model's effective action, and power-law, rather than exponential, correlation functions occur. This is also the main reason why the universality class of this model is different from the Ising model. In general, long-range interactions decrease upper critical dimensions. In that context, as discussed in detail in Ref. [16], the leading order theory is described by the Landau-Ginzburg theory modified for systems with long-range interactions. The non-Gaussian exponents obtained from Eq. (3) are in fact the mean-field exponents for a theory with long-range interactions [16]. Consequently, the scaling region where this "non-Gaussian" scaling is observed is practically infinite, as can be seen from the large-$N$ and computer simulations results [11]: Instead of exhibiting crossover from square root to linear behavior (since $\beta = 1$), the order parameter vanishes linearly over the entire measured region.

The width of the scaling region is determined by the modified Ginzburg criterion [15,16] which states that the Landau-Ginzburg theory breaks down when $|t|^{d-d_c} > 1$, where $d_c$ is the upper critical dimension. In the Gross-Neveu model $d_c$ is determined dynamically and to leading order $d_c = d$. This is why the models are renormalizable for all $d, 2 < d < 4$. Keeping this in mind, it is clear that the Ginzburg criterion is always respected in the sense that deviations from this behavior are never important no matter how close we get to the critical point since $d - d_c = 0$. Subleading corrections in $1/N$ might change this balance, but, as argued in Ref. [10] they are always perturbative, unlike in Yukawa models, where the existence of an additional fixed point $\lambda^*$ could lead to nonperturbative effects. When the temperature is turned on in the Gross-Neveu model, the long-range interactions are screened and traditional mean-field scaling sets in. This is the result we observe in our simulations. Several lattice sizes with large asymmetries were simulated to challenge these ideas.

In conclusion, the study of the Gross-Neveu model suggests that arguments invoking dimensional reduction + universality must be used with care. Our resultsindicate that an effective scalar model fails to describe the Gross-Neveu model at finite temperature. We believe that the reason for this failure in our example is related to the composite nature of the mesons. Pointlike scalars cannot adequately describe the physics in the vicinity of the second transition. The order chiral physical picture behind this failure observes that both the density and the size of the loosely bound $\sigma$ meson increase with temperature. Close to the restoration temperature the system is densely populated with overlapping composites. In other words the fluffiness of the mesons cannot be ignored—the constituent fermions are essential degrees of freedom even in the scaling region, right before the composites dissociate. Similar discussions of the failure of effective meson theories in a slightly different context have been given in Ref. [17].

It is well known that four-fermion models can be used as effective theories of QCD [18]. In addition to having the same global symmetries, the mesons in both theories are composite. Therefore, these models are believed to have common properties over a wide range of scales where the quark substructure of the mesons is relevant. Knowing this, it would be interesting to see what happens in two-flavor QCD [19]: Does it follow the dimensional reduction scenario, or the Gross-Neveu behavior? Of course, these two alternatives do not exhaust all the possibilities [20], but we believe that the scenario suggested by the Gross-Neveu model is sufficiently compelling to warrant further analyses of QCD simulation data.

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