Integrable Systems in Stringy Gravity

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Lax-pair representation is derived for the static axisymmetric Einstein-Maxwell-dilaton and stationary axisymmetric Einstein-Maxwell-dilaton-axion theories in four space-time dimensions.

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Both vacuum and electrovacuum Einstein equations enjoy a complete integrability property being restricted to space-times admitting a two-parameter Abelian group of isometries [1]. This entails rich mathematical structures such as an infinite set of nonlocal conservation laws [2] and Backlund transformations [3]. Similar property is shared by higher-dimensional vacuum Einstein equations independent of the extra coordinates [4]. A large class of scalar-vector gravity coupled systems motivated by Kaluza-Klein (KK) and extended supergravity theories was studied by Breitenlohner, Maison, and Gibbons [5], who listed a variety of integrable models in a nonexplicit form.

Recent interest in such systems is related to the heterotic string theory [6]. A *pure* gravity coupled to a dilaton and an axion was analyzed as a two-dimensionally integrable theory by Bakas [7]. However, the most intriguing features of string-motivated gravity, related to the black hole puzzle, are due to the peculiar nature of the dilaton coupling to *vector* fields [8]. Here we present an explicit proof of integrability of two stringy gravity models including vector fields: the static axisymmetric Einstein-Maxwell-dilaton (EMD) system with an arbitrary dilaton coupling constant, and the stationary axisymmetric Einstein-Maxwell-dilaton-axion (EMDA) system. Although a possibility of a coset representation crucial for integrability can be suspected already from Table I of [5] ($N = 4$ supergravity), an explicit structure of the cosets in terms of the initial four-dimensional variables seems not to have been given so far. Meanwhile, it is necessary in order to derive the corresponding Lax pair and to apply an inverse scattering transform (IST) method [9].

The derivation is similar to one used earlier in the case of vacuum and electrovacuum Einstein equations [10]. It consists of presenting the theory in a space-time possessing a Killing vector field in terms of the three-dimensional sigma model with a subsequent derivation of the *zero-curvature* representation of the equations of motion. The procedure is rather well known, so we just outline its main steps and fix our notation.

Consider a general four-dimensional coupled system of gravitational, U(1) vector, and some scalar massless fields. Assuming the metric to admit a timelike Killing symmetry, one can write the interval as

$$ds^2 = -f(dt - \omega_i dx^i)^2 + f^{-1} h_{ij} dx^i\, dx^j, \qquad (1)$$

where f, ω_i, and the three-metric h_{ij} depend on the space coordinates x^i, $i = 1,2,3$, only. Then the U(1) field is fully describable in terms of electric and magnetic potentials v, a. Usually, in conformity with the Einstein constraints, a twist potential χ may be introduced to generate the rotation one-form ω_i. Together with f and scalars, these variables may be interpreted as a set of scalar fields constituting a source for h_{ij}. If there are no scalar potentials in the initial four-dimensional action, the theory will be equivalent to a three-dimensional sigma model

$$S_\sigma = \frac{1}{2} \int (\mathcal{R} - \mathcal{G}_{AB}(\varphi)\, \partial_i \varphi^A \partial_j \varphi^B h^{ij}) \sqrt{h}\, d^3x\,, \qquad (2)$$

where $\mathcal{R}$ is the three-dimensional scalar curvature, $\varphi^A = (f, \chi, v, a, \text{scalar fields})$, $A = 1, \ldots, K$, and $\mathcal{G}_{AB}$ is the target space metric.

Suppose that the target space is a *symmetric* Riemannian space G/H with N-parameter isometry group G acting transitively on it (H being an isotropy subgroup), generated by the set of N Killing vectors forming the Lie algebra of G, $[K_\mu, K_\nu] = C^\lambda_{\mu\nu} K_\lambda$, $\mu, \nu, \lambda = 1, \ldots, N$. Then the equations of motion for φ^A will be equivalent to the set of conservation laws for Noether currents

$$\partial_i (h^{ij} \sqrt{h} J^\mu_i) = 0, \qquad J^\mu_i = \tau^\mu_A \frac{\partial \varphi^A}{\partial x^i}\,, \qquad (3)$$

built using the corresponding Killing one-forms $\tau^\mu = \eta^{\mu\nu} K^A_\nu \mathcal{G}_{AB} d\varphi^B$, where $\eta^{\mu\nu}$ is an inverse to the Killing-Cartan metric $\eta_{\mu\nu} = k C^\alpha{}_{\mu\beta} C^\beta{}_{\nu\alpha}$. With a proper choice of k these one-forms will satisfy the Maurer-Cartan equation

$$d\tau^\mu + \frac{1}{2} C^\mu{}_{\alpha\beta} \tau^\alpha \wedge \tau^\beta = 0\,. \qquad (4)$$

Let e_μ denote some matrix representation of the Lie algebra of G, $[e_\mu, e_\nu] = C^\lambda{}_{\mu\nu} e_\lambda$. Define the following matrix-valued connection one-form: $\mathcal{A} = \mathcal{A}_B d\varphi^B = e_\mu \tau^\mu$. In view of (4), the corresponding curvature vanishes,

$$\mathcal{F}_{BC} = \mathcal{A}_{C,B} - \mathcal{A}_{B,C} + [\mathcal{A}_B, \mathcal{A}_C] = 0\,, \qquad (5)$$

and thus $\mathcal{A}_B$ is a pure gauge

$$\mathcal{A}_B = -(\partial_B g)\, g^{-1}, \qquad g \in G\,. \qquad (6)$$

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The pullback of $\mathcal{A}$ onto the configuration space x^i is equivalent to (3) and, hence, to the equations of motion of the sigma model. In terms of g Eqs. (3) read

$$d\{(\star dg)\, g^{-1}\} = 0\,, \tag{7}$$

where the star stands for a three-dimensional Hodge dual.

Now impose an axial symmetry condition, representing the three-metric in the Lewis-Papapetrou form:

$$h_{ij}dx^i\, dx^j = e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2. \tag{8}$$

Then (7) becomes equivalent to a modified chiral equation

$$(\rho g_{,\rho} g^{-1})_{,\rho} + (\rho g_{,z} g^{-1})_{,z} = 0\,, \tag{9}$$

and the corresponding Lax pair with a complex spectral parameter λ can be found:

$$D_1\Psi = \frac{\rho U - \lambda V}{\rho^2 + \lambda^2}\Psi, \qquad D_2\Psi = \frac{\rho V + \lambda U}{\rho^2 + \lambda^2}\Psi. \tag{10}$$

Here $V = \rho g_{,\rho} g^{-1}$, $U = \rho g_{,z} g^{-1}$, Ψ is a matrix "wave function," and

$$D_1 = \partial_z - \frac{2\lambda^2}{\rho^2 + \lambda^2}\partial_\lambda, \qquad D_2 = \partial_\rho + \frac{2\lambda^2 \rho}{\rho^2 + \lambda^2}\partial_\lambda \tag{11}$$

are commuting operators; then (9) follows from the compatibility condition $[D_1, D_2]\Psi = 0$. This linearization is sufficient to establish a desired integrability property. The IST method [9] can be directly applied to (10) to generate multisoliton solutions, and an infinite-dimensional algebra of a Geroch-Kinnersley-Chitre (GKC) type can be derived.

Let us apply this formalism to EMD and EMDA systems. The first described by the action

$$S = \frac{1}{16\pi}\int [R - 2(\partial\phi)^2 - e^{-2\alpha\phi}F^2]\sqrt{-g}\, d^4x\,, \tag{12}$$

where ϕ is the real scalar field (dilaton), $F = dA$ is the Maxwell two-form, and α is the dilaton coupling constant. For $\alpha = 0$, (12) reduces to the Brans-Dicke-Maxwell (BDM) action in the Einstein frame (with the Brans-Dicke parameter $\omega = -1$). For $\alpha = \sqrt{3}$, (12) is derivable from the five-dimensional KK theory.

In conformity with the Maxwell equations following from (12), electric and magnetic potentials can be introduced via

$$F_{i0} = \frac{1}{\sqrt{2}}\partial_i v, \qquad F^{ij} = -\frac{f}{\sqrt{2h}}e^{2\alpha\phi}\epsilon^{ijk}\partial_k a\,, \tag{13}$$

while the twist potential χ is defined through

$$\tau_i = \partial_i\chi + v\partial_i a - a\partial_i v, \qquad \tau^i = f^2\epsilon^{ijk}\partial_j\omega_k/\sqrt{h} \tag{14}$$

(three-dimensional indices are raised and lowered using h_{ij}). The corresponding target space is five-dimensional ($K = 5$), and

$$\mathcal{G} = \frac{1}{2f^2}[df^2 + (d\chi + v da - a dv)^2] - \frac{1}{f}(e^{-2\alpha\phi}dv^2 + e^{2\alpha\phi}da^2) + 2d\phi^2. \tag{15}$$

For $\alpha = 0$ and $\phi = $ const this metric reduces to one given by Neugebauer and Kramer for the Einstein-Maxwell (EM) system [10].

For a general stationary class of metrics (1), the target space (15) is a symmetric Riemannian space only for $\alpha = 0, \sqrt{3}$, when it has the structure of cosets SU(2,1)/S[U(2) × U(1)] × R and SL(3,R)/SO(3), respectively, corresponding to the BDM and five-dimensional KK theories [5]. For $\alpha \neq 0, \sqrt{3}$ the isometry group of (15) is only $N = 5$ solvable subgroup of SL(3,R). However, if an additional condition of *staticity* is imposed, $\omega_i = 0$, the (truncated) target space turns out to be a symmetric space for *arbitrary* α. For the static geometry it is consistent to consider electric and magnetic configurations separately. Both will be described by the same equations after reparametrization

$$\xi = (\alpha\phi - \tfrac{1}{2}\ln f)/\nu, \qquad \eta = [\phi + (\alpha/2)\ln f]/\nu\,, \tag{16}$$

for a magnetic case, and

$$\xi = -(\alpha\phi + \tfrac{1}{2}\ln f)/\nu, \qquad \eta = [\phi - (\alpha/2)\ln f]/\nu\,, \tag{17}$$

for an electric one, where $\nu = (\alpha^2 + 1)/2$. Denoting as u either magnetic (a) or electric (v) potentials, respectively, one can present the line element of the truncated target space as $dl_3^2 = d\eta^2 + dl_2^2$, where

$$dl_2^2 = d\xi^2 - e^{2\nu\xi}du^2. \tag{18}$$

This two-dimensional space can be easily shown to represent a coset SL(2,R)/U(1). Indeed, one can find three Killing vectors for (18),

$$K_1 = \partial_u, \qquad K_2 = p\partial_u - \nu^{-1}u\partial_\xi, \qquad K_3 = u\partial_u - \nu^{-1}\partial_\xi\,, \tag{19}$$

where $p = (u^2 + \nu^{-2}e^{-2\nu\xi})/2$, with the sl(2,$R$) structure constants ${C^3}_{12} = {C^2}_{32} = {C^1}_{13} = 1$. The corresponding Killing-Cartan one-forms, with the normalization $k = (2\nu)^{-2}$, will satisfy (4), and $dl_2^2 = \frac{1}{2}\eta_{\mu\nu}\tau^\mu \otimes \tau^\nu$, where $\eta_{\mu\nu} = 2k\,\mathrm{diag}(1, 1, -1)$. Choosing as e_μ a 2×2 representation of sl(2,R), one can find from (6) the following matrix $g \in$ SL(2,R)/U(1):

$$g = \nu e^{\nu\xi}\sqrt{2}\begin{pmatrix} u^2 - p & -u/\sqrt{2} \\ -u/\sqrt{2} & 1 \end{pmatrix}. \tag{20}$$

Alternatively, in view of the isomorphism SL(2,R) ~ SO(2,1), a 3×3 representation in terms of SO(2,1)/SO(2) coset can be derived. In the axisymmetric case both can be used in the Lax equations (10).

For $\alpha = 0$ ($\nu = \frac{1}{2}$) the above theory reduces to the corresponding representation for electrovacuum. Since, as we have shown, the underlying algebraic structure is α independent, already this fact is sufficient to reveal integrability of the static axisymmetric EMD system with arbitrary α. However, in the *stationary* case the coset representation does not exist for the arbitrary-α EMD system.

Remarkably, the EMDA theory turns out to be more symmetric and is integrable in the *stationary* axisymmetric case too. The EMDA action in four dimensions reads

$$S = \frac{1}{16\pi} \int \Big\{ R - 2\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} e^{4\phi} \partial_\mu \kappa \partial^\mu \kappa - e^{-2\phi} F_{\mu\nu} F^{\mu\nu} - \kappa F_{\mu\nu} \tilde{F}^{\mu\nu} \Big\} \sqrt{-g}\, d^4x \,, \tag{21}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2} E^{\mu\nu\lambda\tau} F_{\lambda\tau}$, κ is an axion field. An electric potential is still introduced through the first of Eqs. (13), while for a one has

$$e^{-2\phi} F^{ij} + \kappa \tilde{F}^{ij} = -f \epsilon^{ijk} \partial_k a / \sqrt{2h} \,. \tag{22}$$

For a twist potential (14) remains valid. The target space now is six dimensional ($K = 6$), and its metric reads

$$\mathcal{G} = \frac{1}{2} e^{-4\phi} \omega_\kappa^2 + 2 d\phi^2 + \frac{1}{2} \left(\frac{df^2}{f^2} + f^2 \omega_\chi^2 \right) - f \{ e^{2\phi} \omega_v^2 + e^{-2\phi} \omega_a^2 \} , \tag{23}$$

where

$$\omega_\kappa = e^{4\phi} d\kappa , \qquad \omega_\chi = f^{-2} (d\chi + v da - a dv) ,$$
$$\omega_v = -f^{-1} e^{-2\phi} dv , \qquad \omega_a = -f^{-1} e^{2\phi} (da - \kappa dv) . \tag{24}$$

Note, that the EMDA theory does not include the EMD one as a particular case. Indeed, setting $\kappa = 0$ gives a constraint $F\tilde{F} = 0$. Similarly, the EMD theory does not contain the EM one: setting $\phi = 0$ gives another constraint $F^2 = 0$.

As it was shown recently [11], the space (23) possesses a $N = 10$ isometry group consisting of scale, three-gauge, three axion-dilaton duality, and three Ehlers-Harrison-type transformations, which unify T and S string dualities in the four-dimensional zero-slope heterotic string theory. Here we will show that the target space is a *symmetric* space which can be identified with the coset SO(3,2)/[SO(3) × SO(2)]. (This coset seems to be reminiscent of the SO(8,2)/[SO(8) × SO(2)] coset mentioned in [5] in the context of the $N = 4$ supergravity in the case of only one vector field nonvanishing.) Denoting generators of SO(3,2) by pair indices $ab, a < b$, where $a, b = 0, \theta, 1, 2, 3$ correspond to the invariant metric $G_{ab} = \mathrm{diag}(-1, -1, 1, 1, 1)$, one has

$$[M_{ab} M_{cd}] = G_{bc} M_{ad} - G_{ac} M_{bd} + G_{ad} M_{bc} - G_{bd} M_{ac} \,. \tag{25}$$

The set of ten one-form satisfying Maurer-Cartan equations with the structure constants $C^{cd}{}_{ab\,ef}$ from (25) reads as follows. An Abelian subalgebra of so(3,2) corresponds to

$$-\tau^{01} = \omega_1 + \omega_f , \qquad \tau^{\theta 2} = \omega_1 - \omega_f - 2\omega_2 , \tag{26}$$

where

$$\omega_1 = \kappa \omega_\kappa - 2d\phi + a(v \omega_\chi + 2\omega_a) ,$$
$$\omega_f = f^{-1} df + \chi \omega_\chi , \qquad \omega_2 = v \omega_v + \tilde{a} \omega_a \,. \tag{27}$$

We introduce a recurrent sequence

$$\begin{aligned} &\omega_3 = \kappa \omega_a - \omega_v , \qquad \omega_4 = a \omega_\chi + \omega_3 , \\ &\omega_5 = v \omega_\chi + \omega_a , \\ &\omega_6 = d\kappa - \kappa^2 \omega_\kappa + 4\kappa d\phi - a(\omega_4 + \omega_3) , \\ &\omega_7 = \omega_\kappa + v(\omega_a + \omega_5) , \\ &\omega_8 = a \tau^{01} - v \omega_6 - \chi \omega_3 + a \omega_2 + da , \\ &\omega_9 = v \tau^{\theta 2} - a \omega_7 - \chi \omega_a + v \omega_2 + dv , \\ &\omega = a \omega_9 - v \omega_8 + \chi(\chi \omega_\chi - \omega_2 - 2\omega_f) + d\chi \,. \end{aligned} \tag{28}$$

Then the remaining set will read

$$\begin{aligned} &2\tau^{0\theta} = \omega + \omega_6 - \omega_7 - \omega_\chi , \\ &2\tau^{02} = \omega - \omega_6 - \omega_7 + \omega_\chi , \\ &-2\tau^{\theta 1} = \omega + \omega_6 + \omega_7 + \omega_\chi , \\ &2\tau^{12} = \omega - \omega_6 + \omega_7 - \omega_\chi , \\ &-\tau^{03} = \omega_5 + \omega_8 , \qquad \tau^{13} = \omega_5 - \omega_8 , \\ &\tau^{\theta 3} = \omega_4 - \omega_9 , \qquad -\tau^{23} = \omega_4 + \omega_9 \,. \end{aligned} \tag{29}$$

In terms of τ^{ab} one has $\mathcal{G}_{AB} = \frac{1}{2} \eta_{ab\,cd} \tau_A^{ab} \tau_B^{cd}$, where $\eta_{ab\,cd} = \frac{1}{12} C^{gh}{}_{ab\,ef} C^{ef}{}_{cd\,gh}$.

Now, using an adjoint representation of so(3,2), one can build a 5 × 5 connection one-form $\mathcal{A}$ and the corresponding matrix $g \in \mathrm{SO}(3,2)/[\mathrm{SO}(3) \times \mathrm{SO}(2)]$. Fortunately, due to isomorphism $\mathrm{SO}(3,2) \sim \mathrm{Sp}(4,R)$, there exists also more concise representation in terms of 4 × 4 matrices. The symplectic connection reads

$$\mathcal{A} = \begin{pmatrix} C & D \\ F & -C^T \end{pmatrix} , \tag{30}$$

where D, F, C are 2 × 2 matrices, $D^T = D$, $F^T = F$,

$$\begin{aligned} C &= \tfrac{1}{2} \{ \tau^{03} I_2 - \tau^{\theta 2} \sigma_x - i \tau^{12} \sigma_y + \tau^{\theta 1} \sigma_z \} , \\ D &= \tfrac{1}{2} \{ (\tau^{0\theta} - \tau^{\theta 3}) I_2 + (\tau^{23} - \tau^{02}) \sigma_x + (\tau^{01} + \tau^{13}) \sigma_z \} , \\ F &= \tfrac{1}{2} \{ -(\tau^{0\theta} + \tau^{\theta 3}) I_2 - (\tau^{23} + \tau^{02}) \sigma_x + (\tau^{01} + \tau^{13}) \sigma_z \} . \end{aligned} \tag{31}$$

Here I_2 is a unit matrix and $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices with σ_z diagonal. In view of (4), the equations of motion of the EMDA sigma model are equivalent to vanishing of the curvature (5) related to (30). This implies the existence of the symmetric symplectic 4 × 4 matrix $g \in \mathrm{Sp}(4,R)/\mathrm{U}(2)$ entering Belinskii-Zakharov representation.

To summarize, we have shown that target space corresponding to the static EMD with an arbitrary dilaton coupling and the stationary EMDA systems in four dimensions are symmetric Riemannian spaces isomorphic to cosets $SO(2,1)/SO(2)$ and $SO(3,2)/[SO(3) \times SO(2)]$, respectively. This ensures a possibility of zero-curvature representation of the equations of motion and an existence of the Lax pair in the axisymmetric case. The explicit expressions for matrices in terms of the initial variables open a way for further application of the IST method. Current algebras associated with $SL(2,R)$ and $Sp(4,R)$ generate infinite-dimensional GKC-type symmetries of the zero-slope heterotic string effective action. Obviously, the whole reasoning can be generalized to the case of a spacelike initial Killing vector field, as well as to the case of the Euclidean signature of the four-space.

As it was noted in [11], the isometry group of the EMDA target space is larger than the product of well-known T and S string dualities [12]. Now it is clear that, on the class of space-times admitting a two-parameter Abelian isometry group, both of these symmetries are particular elements of the infinite-dimensional GKC-type group. The implications of this to the *exact* string theory are still to be explored. An intriguing question is whether classical integrability of the two-dimensional reduced EMDA system entails the possibility of an explicit construction of new classes of exact string backgrounds in terms of the gauged WZW models. This issue will be discussed in a subsequent publication.

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