# Classical Squeezing of an Oscillator for Subthermal Noise Operation

Vasant Natarajan,* Frank DiFilippo, and David E. Pritchard
*Research Laboratory of Electronics, Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139*
(Received 25 July 1994)

We have demonstrated that, analogous to the use of quantum mechanical squeezing to beat the quantum shot noise limit, classical squeezing can be used to reduce the effect of thermal amplitude noise in precision measurements. The classical oscillator under study is a single ion in a Penning trap whose motion is cooled by coupling to a thermal bath. The thermal noise distribution is quadrature squeezed by parametric amplification. We observe 6 dB of noise reduction below the equilibrium value when the squeezing axis is 90° out of phase with the coherent signal drive.

PACS numbers: 06.20.–f, 05.40.+j, 07.75.+h, 46.30.Rc

Squeezed states of the photon field are currently receiving wide attention as a means of reducing noise in optical interferometers and communication networks [1]. This noise arises from the Heisenberg uncertainty principle which sets a fundamental limit to the simultaneous knowledge of conjugate observables such as the number and phase of the photon field. Coherent states have equal uncertainty in the two observables, while squeezed states have the uncertainty redistributed such that one variable has reduced uncertainty at the expense of increased uncertainty in the other. By ensuring that the signal is present in the variable with the smaller uncertainty, one can beat the quantum mechanical shot noise limit.

Although nearly all squeezing experiments to date have involved light, the use of squeezing is not restricted to quantum systems. In classical systems, analogous squeezing methods can redistribute noise of thermal origin [2,3]. While there is no fundamental limit to the product of variances of conjugate phase space variables in classical analysis, thermal fluctuations cause an ensemble of possible initial states to have a Gaussian distribution in phase space the same as a minimum uncertainty quantum state. In thermal equilibrium, the variance in the two (properly normalized) conjugate variables is identical, and the distribution of initial states is circularly symmetric. After squeezing, the distribution becomes noncircular, as was first shown by Rugar and Grütter for a parametrically driven mechanical oscillator [2]. In this paper, we (quadrature) squeeze the thermal motion of another classical harmonic oscillator—a single ion in a Penning trap. We show that the effect of thermal amplitude fluctuations, which can limit a precision measurement of the oscillation frequency, are significantly reduced by using a squeezed initial distribution. To our knowledge, this is the first use of squeezing on a classical system to beat the thermal noise limit on a measurement. This approach may be of general applicability as a novel technique to solve the problem of thermal noise in measurements on mechanical systems, in both the classical and quantum regimes. We have also proposed [3] ways of producing amplitude squeezed states which may result in better noise reduction than demonstrated here.

In order to understand our study, one needs to consider the dynamics of a single ion in a Penning trap in some detail. The ideal Penning trap [4] consists of a strong uniform magnetic field and a weak quadrupole electric field. The motion of an ion then decomposes into three normal modes: an axial mode (at $\omega_z$) along the magnetic field axis and two radial modes perpendicular to the field—a rapid cyclotron motion (at $\omega_c'$) and a slow $\mathbf{E} \times \mathbf{B}$ magnetron drift (at $\omega_m$). In a real trap, there are always residual deviations from the ideal trapping fields which cause anharmonicities in the potentials.

We use our Penning trap as a precision mass spectrometer [5]. Hence, we are primarily interested in measuring the trap oscillation frequencies to high precision in a short time. We detect only the axial motion of the ion directly. We use a high $Q$ ($\sim$25 000) superconducting detector operating around 160 kHz, immersed in a 4 K liquid He bath [6]. The process of detection draws energy out of the ion's motion (giving a damping time on the order of a few seconds) until it comes to thermal equilibrium with the noise of the detector. To access the radial modes, we couple them to the axial mode using a coupling pulse (called a "$\pi$ pulse") that exchanges both phase and (suitably normalized) amplitudes between the coupled modes [7]. In this way the (squeezed) thermal distribution of the axial motion can be transferred to the radial mode.

The anharmonicities in the electric and magnetic fields, mentioned earlier, combined with the residual thermal amplitude in the modes is at the heart of our problem. The anharmonicities make the frequency amplitude dependent so that thermal fluctuations in the amplitude appear as frequency noise during a precision measurement. Actually, even with ideal fields, the frequency of any classical oscillator is amplitude dependent because of the special relativistic mass shift. (This "anharmonicity" is particularly important for the cyclotron mode whose frequency is precisely measured in mass spectrometry.) To lowest order, all these anharmonicities cause the frequency of any mode

0031-9007/95/74(15)/2855(4)$06.00  

to vary as the square of its amplitude, as is illustrated in the following expression for the dependence of the axial frequency on the axial amplitude ($a_z$) [4]:

$$\frac{\delta\omega_z}{\omega_z} = \left(\frac{3C_4}{4d^2} - \frac{\omega_z^2}{2c^2}\right)a_z^2, \qquad (1)$$

where $C_4$ is the lowest order electrostatic correction, $d$ is the size of the trap, and $c$ is the speed of light.

Squeezing techniques can alleviate these frequency shifts by reducing thermal variations of the amplitude. We squeeze the thermal motion using parametric amplification, the most common method of quadrature squeezing, which is effected in the axial mode by modulating the trapping potential at twice the axial frequency. To see how this comes about, we consider the equation of motion for the axial mode in the presence of a modulating drive,

$$\ddot{z} - \omega_z^2(1 + \varepsilon\cos 2\omega_z t)z = 0, \qquad (2)$$

where $\varepsilon$ parametrizes the strength of the parametric drive. We assume that the motion is perfectly harmonic and undamped. The solution to this equation can be separated into in-phase and out-of-phase quadratures as

$$z(t) = C(t)\cos\omega_z t + S(t)\sin\omega_z t, \qquad (3)$$

where $C(t)$ and $S(t)$ are slowly varying compared to $\omega_z$. Their time evolution is given by the well-known dependences $C(t) = C(0)e^{\varepsilon\omega_z t/4}$ and $S(t) = S(0)e^{-\varepsilon\omega_z t/4}$ [8]. Therefore, the in-phase component is amplified while the out-of-phase component is attenuated (hence the name quadrature squeezing), both by the same factor $S_F$,

$$S_F = e^{\varepsilon\omega_z t/4}, \qquad (4)$$

thereby maintaining phase space density (in accordance with Liouville's theorem).

The simplifying assumptions we have made for this analysis are actually not too restrictive. The assumption of undamped motion remains valid as long as the squeezing pulse lasts for a time much shorter than the thermalization time. Although we could have increased the thermalization time by detuning the ion far from the high-$Q$ detector's response, for the experiments reported here we left the ion on resonance and used a 100 ms squeezing pulse, which is small compared to the ion's 3.3 s damping time. We are also justified in neglecting anharmonicity in the axial response during squeezing because the relative change in the axial frequency is less than $10^{-6}$ as long as the ion samples are only 15% of the trap's axial extent. The thermal rms amplitude is 1.5% of the trap size, and we restricted the squeezing factor $S_F$ to 3, thus this approximation is valid.

To make a frequency measurement on a mode, we excite its amplitude to a finite value using a coherent drive pulse. This adds vectorially to the preexisting thermal noise distribution and creates an analog of the quantum coherent state, as shown in Fig. 1. The two quadrature components of the thermal noise now appear as amplitude and phase noise. When the noise is squeezed, the

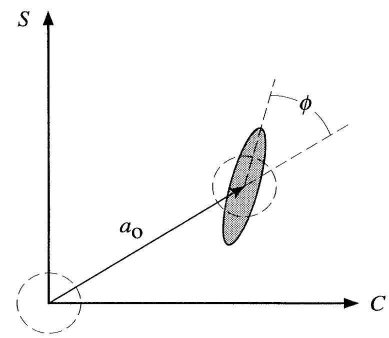


FIG. 1. Squeezed distribution produced by parametric amplification. The circularly symmetric thermal noise distribution (dotted) at the origin evolves into an elliptic distribution (solid) along hyperbolic trajectories when the squeezing pulse is applied. When the excitation pulse is added, the final amplitude uncertainty depends on the phase difference between the two pulses.

amplitude variation after the addition of the excitation pulse depends on both the squeezing factor $S_F$ and the relative phase $\phi$ between the excitation pulse and the squeeze pulse. The variance in the square of the amplitude (since our frequency noise goes as $a^2$) is given by

$$\mathrm{var}[a^2] = 2a_0^2 a_{\mathrm{th}}^2\left(S_F^2\cos^2\phi + \frac{1}{S_F^2}\sin^2\phi\right) + \frac{a_{\mathrm{th}}^4}{2}\left(S_F^4 + \frac{1}{S_F^4}\right), \qquad (5)$$

where $a_{\mathrm{th}}$ is the thermal rms amplitude and $a_0$ is the excitation amplitude. We can recover the variance for the thermal equilibrium distribution by setting $S_F = 1$. For small squeezing factors, the maximum improvement over the equilibrium case occurs when $\phi = \pi/2$. Clearly, increasing the squeezing beyond a certain point is not advantageous because the ends of the squeezed distribution increase the variance of the amplitude significantly—this is reflected in the $a_{\mathrm{th}}^4$ term [9]. The optimum value of $S_F$ depends on the ratio $a_0/a_{\mathrm{th}}$. Under our experimental conditions, this ratio was about 10, for which $S_F = 2.4$ would produce the greatest improvement in $\mathrm{var}[a^2]$.

Since we squeeze the axial motion, we would ideally like to apply the excitation pulse also to the axial motion. In practice, a sufficiently large excitation pulse would cause the axial anharmonicity to be too large for us to determine the amplitude and phase accurately. Instead, we transfer the squeezed thermal distribution from the axial mode to the cyclotron mode using our standard $\pi$-pulse technique [7], and then add the excitation pulse, producing squeezed cyclotron motion. We then measure the cyclotron amplitude $\rho_c$ using the dependence of the frequency of the axial mode on the cyclotron amplitude [4]

$$\frac{\delta\omega_z}{\omega_z} = \frac{\omega_c'}{\omega_m}\frac{B_2}{4B_0}\rho_c^2, \tag{6}$$

where $B_2/B_0$ is the second order correction to the magnetic field. This method is well known and has been used for detecting cyclotron resonance before [10].

The basic plan of our squeezing demonstration is as follows. We thermalize the axial motion of the ion by coupling it to our detector. We then use parametric amplification to squeeze one quadrature component of the axial motion at the expense of the other. This squeezed noise is transferred to the cyclotron mode and displaced by a fixed amount using a pulse whose phase angle is varied with respect to the axis of the squeezing, as in Fig. 1. This causes the squeezed noise to appear as amplitude and phase noise of the cyclotron motion. The cyclotron amplitude is then determined by measuring the shift in the frequency of the axial oscillation, and its variance is determined by repeating the whole procedure many times. The squeezing produces an oscillatory dependence of the variance on the relative phase angle.

For the experiments we used a single $Ne^+$ ion in the trap. The normal mode frequencies for $Ne^+$ were 160 kHz (axial), 6.5 MHz (cyclotron), and 2 kHz (magnetron). Before squeezing, the rms thermal amplitude in the axial mode was $\sim 100\ \mu$m. During transfer to the cyclotron mode, this thermal amplitude was transformed to 15 $\mu$m; we then applied a cyclotron excitation amplitude of 170 $\mu$m. The magnetic bottle was $B_2/B_0 \approx 10^{-5}\ \mathrm{cm}^{-2}$ [11]. During each measurement sequence, we measured the axial frequency with and without the cyclotron excitation pulse. The measurement without the excitation pulse represents a "zero-$\rho_c$" axial frequency, and thus has no $B_2$ shift. We used this as a reference to remove any drift in the axial frequency due to drift in the trap dc voltage that determines $\omega_z$.

The variance in the axial frequency was estimated by repeating this sequence 100 times for each setting of $S_F$ and $\phi$. The error in this estimate was found by assuming that the set of $N$ values thus obtained is a random sample drawn from a normal distribution [12]. For $N = 100$, such a sample gives an estimate of the variance with 12% relativer error, which is sufficient for our purposes. In addition to the error in the measured values of the variance due to the finite sample size, technical noise contributes a constant background representing about 25% of the noise from unsqueezed thermal fluctuations. This arises from uncertainty in the determination of the axial frequency from the detected signal as well as intrinsic noise in the frequency due to fluctuations in the trap voltage. These variances generate a constant baseline which was subtracted from the measured noise to obtain the thermal noise.

The results of axial frequency measurements for a squeezed cyclotron amplitude distribution are shown in Fig. 2. We kept the squeezing factor constant by using a fixed strength for the squeezing drive while we varied its relative phase. Because of unknown transfer functions, the phase and amplitude of the drive at the ion's location was not known *a priori* and had to be determined by fitting Eq. (5) to the data. We have also done experiments where we varied the squeezing drive strength with a constant phase. From those results, we were able to estimate the drive strength needed to obtain an optimal squeezing factor for our experimental parameters. From the data in Fig. 2, the best fit by Eq. (5) is obtained with a squeeze factor of 2.6, close to the optimum value. Under these conditions, the maximum reduction in the measured noise in $\omega_z$ is about 6 dB below the thermal equilibrium value.

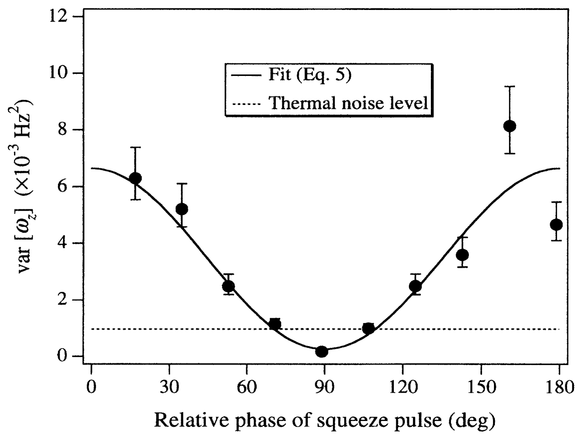


FIG. 2. Variance in axial frequency. The axial frequency noise was measured as a function of the relative phase of the squeeze pulse while the strength of the parametric drive was kept fixed. The solid line is a best fit by Eq. (5) (from text) and was obtained using a squeeze factor of 2.6. The maximum reduction below the equilibrium value is 6 dB and occurs when the phase difference is $\pi/2$.

We could also use squeezing to benefit our measurement of the oscillator frequency in other ways. When the relative phase angle is 0, the initial state of the oscillator is "phase squeezed." Since we measure the frequency by measuring the phase accumulated in a given time, we can obtain higher precision in a shorter time if the initial thermal phase noise is reduced. The measurement of the phase thereby accumulated could also be limited by technical noise. In such a case, we can use the parametric drive as a phase amplifier to "spread out" the phase around a mean value and reduce the effect of measurement uncertainty. This approach will work as long as amplitude-dependent effects do not limit our precision.

In conclusion, we have demonstrated that it is possible to squeeze the thermal distribution of a classical oscillator to reduce its amplitude fluctuations and its effect on a frequency measurement. In the work reported here, we operated in the classical limit due to the high temperature of the oscillator. However, using laser cooled ions, it is possible to cool to the ground state of the ion's motion. Our squeezing procedure would then produce quantum mechanical squeezed states. These states are of interest in reducing noise during spectroscopic measurements on trapped ions [13].

This work was supported by NSF Grant No. 9222768-PHY and Joint Services Electronics Program Grant No. DAAL03-92-C-0001. One of us (F. D.) acknowledges additional support from an NSF Graduate Fellowship.

*Current address: AT&T Bell Laboratories, Murray Hill, NJ 07974.

[1] R.E. Slusher *et al.,* Phys. Rev. Lett. **55**, 2409 (1985); L. Wu *et al.,* Phys. Rev. Lett. **57**, 2520 (1986).
[2] D. Rugar and P. Grütter, Phys. Rev. Lett. **67**, 699 (1991).
[3] F. DiFilippo, V. Natarajan, K.R. Boyce, and D.E. Pritchard, Phys. Rev. Lett. **68**, 2859 (1992).
[4] L. Brown and G. Gabrielse, Rev. Mod. Phys. **58**, 233 (1986).
[5] E.A. Cornell *et al.,* Phys. Rev. Lett. **63**, 1674 (1989); F. DiFilippo, V. Natarajan, and D.E. Pritchard (to be published).
[6] R.M. Weisskoff *et al.,* J. Appl. Phys. **63**, 4599 (1988).
[7] E.A. Cornell, R.M. Weisskoff, K.R. Boyce, and D.E. Pritchard, Phys. Rev. A **41**, 312 (1990).
[8] L.D. Landau and E.M. Lifshitz, *Mechanics* (Pergamon Press, Oxford, 1976), 3rd ed.
[9] This limit on the improvement is inherent to the technique of quadrature squeezing. We have proposed more complicated schemes where we use the anharmonicity of the oscillator to overcome this limit and produce true amplitude squeezing (see Ref. [2]).
[10] R.S. Van Dyck, Jr., P.B. Schwinberg, and H.G. Dehmelt, Phys. Rev. Lett. **59**, 26 (1987).
[11] This value of $B_2/B_0$ is about an order of magnitude higher than the value to which we can shim this magnetic field imperfection. However, we left it at this high value in order to reduce the time for measuring the cyclotron amplitude, since the shift in the axial frequency was correspondingly higher.
[12] H.D. Brunk, *An Introduction to Mathematical Statistics* (Ginn, Boston, 1960).
[13] D.J. Heinzen and D.J. Wineland, Phys. Rev. A **42**, 2977 (1990).