Effective Masses of Ions in Superfluid ^{3}He-B

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We show that ion masses in superfluid ^{3}He ought to be enormously enhanced (by a factor of 10^2) as compared with the same ion masses in ^{4}He measured at low temperature. We calculate precisely the dependence of the effective mass on pressure in ^{3}He-B, and show that the coherent (ballistic) motion of ions in ^{3}He-B can be studied experimentally at $T < (0.3-0.2)T_c$.

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The problem of ion motion in normal ^{3}He liquid has been of long-standing interest, partly because of its connection with the "orthogonality catastrophe," but mostly because theorists have had a hard time explaining it. The ion motion is greatly overdamped at low temperature, by multiple scattering of ^{3}He quasiparticles, so theorists have concentrated on calculating the experimentally measurable ion mobility. Early perturbative calculations [1] predicted a mobility $\mu(T)$ diverging as $1/T^2$ below a temperature $T_0 = p_F^2/M$, where M is the bare ion effective mass [$M \sim (100-260)m_3$, depending on pressure, for the negative ion; here m_3 is the ^{3}He atomic mass]. Experiments on both positive [2–4] and negative [2,5–8] ions flatly contradicted this prediction; μ_- is roughly constant through and below T_0, all the way down to the superfluid transition T_c.

However, this problem is a strong-coupling one. The dimensionless ion-^{3}He coupling is $g = p_F^2\sigma_{\rm tr}/3\pi^2$, with $\sigma_{\rm tr}$ the transport cross section, and $g \gg 1$. The high-T scattering rate equals $\Gamma = T_0 g \gg T_0$, which is why the ion motion is overdamped already for $T \gg T_0$. Moreover, it was realized by Josephson and Lekner [9] that for $T < \Gamma$ the ion recoil is not free, but Brownian diffusive, down to the unobservably low temperature $T_{\rm coh} = T_0 g e^{-g}$. This diffusive motion means that it is meaningless to define an effective mass for the ion above $T_{\rm coh}$. The theory of ion mobility in normal ^{3}He has nevertheless been considerably refined since then [10,11].

One obvious way for experimentalists to see coherent motion of an ion in ^{3}He is to go to the superfluid phases, where the gap cuts off the orthogonality catastrophe [10]. Remarkably, this possibility has not been explored, neither in theory nor experiment (although some mobility theory and experiments have been done [8]—we return to these below). In this paper I give detailed theory of ion dynamics, which is exact in the large-g limit. A very striking prediction emerges from this analysis—that the effective mass of ions in the superfluid phases will be very large (up to $2 \times 10^4 m_3$, or some 100 times the bare ion mass). I calculate the effective mass $M^{\rm eff}(P)$ as a function of pressure in the low-T limit in ^{3}He-B, and suggest how this prediction might be verified experimentally. This prediction (which is clearly out of the framework of the standard models [12]) should constitute a very stringent test of our ideas of particle dynamics in a Fermi liquid.

The Hamiltonian is that of a spherical object in a Fermi liquid environment:

$$H = \frac{1}{2} M\mathbf{R}^2 + H_F + V\,,$$

$$V = \int d\mathbf{r}\, V(\mathbf{r} - \mathbf{R})\hat{\rho}(\mathbf{r})\,, \qquad (1)$$

where $\mathbf{R}$ is the ion coordinate, H_F the Hamiltonian of ^{3}He, and $\hat{\rho}(\mathbf{r})$ the ^{3}He density operator. We make use of the path integral technique and integrate out the fermion degrees of freedom [11], and start by considering the case of normal ^{3}He. Using Feynman's path integral over $\mathbf{R}$ in imaginary time [13] the effective action in the partition function can be written

$$S = S_0 - \iint_0^\beta d\tau\, d\tau' \,\frac{\pi}{\beta} \sum_{n\neq 0} \mathcal{F}_n(\mathbf{R}_\tau - \mathbf{R}_{\tau'})e^{i\omega_n(\tau-\tau')}, \qquad (2)$$

where $S_0 = \int d\tau\, M\dot{\mathbf{R}}^2/2$, $\omega_n = 2\pi n/\beta$ are Matsubara frequencies, and the influence functional $\mathcal{F}_n(\mathbf{R})$ is related to the overlap integral between the initial and final Fermi liquid states with different local potentials [11]; one has

$$\mathcal{F}_n(\mathbf{R}) = \frac{|\omega_n|}{16\pi^2}\,\mathrm{Tr}\{\ln^2(S_f S_i^{-1})\}. \qquad (3)$$

where $S_f = S(\mathbf{R})$ and $S_i = S(0)$ are the scattering S matrices at the Fermi energy in the final (the particle at the point $\mathbf{R}$) and initial ($\mathbf{R} = \mathbf{0}$) states. The connection with the overlap integral $\langle f \mid i\rangle$ is established by

$$\ln|\langle f \mid i\rangle| = 2\int_0^\infty \frac{d\omega\, \coth(\omega/2T)}{\omega^2}\,\mathrm{Im}F^R(\omega, \mathbf{R})\,. \qquad (4)$$

The effective action (2) and (3) is correct provided we deal with heavy particles, $M \gg m_3$.

The formal expression (3) is highly nonlinear in R and cannot be solved in general. However, in the strong-coupling limit $g \gg 1$ we can restrict ourselves to a quadratic expansion

$$\mathcal{F}_n(\mathbf{R}) = \frac{g|\omega_n|}{4\pi}\,(p_F R)^2, \qquad (5)$$

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which results in a simple quadratic action

$$S^{(2)} = \frac{M\beta}{2} \sum_n (\omega_n^2 + \Gamma|\omega_n|)|\mathbf{R}_n|^2, \qquad (6)$$

where $\mathbf{R}_\tau = \sum_n \mathbf{R}_n e^{i\omega_n \tau}$. Moreover, if we calculate the mean square value of the particle displacement using Eq. (6),

$$p_F^2 \langle (\mathbf{R}_\tau - \mathbf{R}_{\tau'})^2 \rangle \approx \frac{3}{\pi g} \ln \frac{\Gamma}{2\pi T}, \qquad (T \ll \Gamma),$$

we find [11,14] that the expansion (5) is justified in the normal state down to T_{coh}; the higher order terms in the expansion $(p_F R)^2 - C_4 (p_F R)^4 + \cdots$ give rise to small corrections proportional to $(20/\pi g) C_4 \ln \frac{\Gamma}{T}$ which can be neglected at $T \gg T_{\mathrm{coh}}$. The case of negative ions is of most importance here because for the hard sphere potential with $p_F R_- \gg 1$ (R_- is the bubble radius) the coefficient C_4 turns out to be very small, $C_4 \sim 10^{-2}$–10^{-3}.

Now in the *normal state,* the mobility μ is given in linear response, and in the R^2 approximation, by

$$\mu/e = \frac{i\omega}{M\omega^2 - f(\omega)} \equiv \frac{1}{M} \frac{1}{-i\omega + \Gamma}, \qquad (7)$$

where $f(\omega) = 4\pi F^R(\omega)/R^2$, and e is the particle charge. While this describes diffusive motion for $\omega < \Gamma$, one can also think of Eqs. (6) and (7) as describing a frequency-dependent mass renormalization $M^{\mathrm{eff}}/M = 1 + \Gamma/|\omega_n|$. In the diffusive regime one assumes that $M^{\mathrm{eff}} \langle v^2 \rangle \propto T$; then Einstein's relation $\mu \propto \langle v^2 \rangle \tau / T$, with τ the scattering time, plus the experimentally observed $\mu(T) = \mathrm{const}$, leads to the conclusion that $M^{\mathrm{eff}} \propto 1/T$, since certainly $\tau \geq 1/T$. This, however, is only very indirect evidence for a temperature-dependent effective mass.

We therefore consider the ion motion in the *superfluid state.* To do this we must consider the overlap integral (4), and the starting point may be the expression derived by Yamada and Yosida [15] at $T = 0$ for the normal state which can be readily generalized for the case of finite temperature superfluid [16]. However, we need only the lowest order term in R^2 here, which allows us to treat the difference between the initial and final state Hamiltonians

$$\Delta V_{\mathbf{p}\mathbf{p}'} = i(\mathbf{p} - \mathbf{p}') \cdot \mathbf{R}(V)_{\mathbf{p}\mathbf{p}'} \qquad (8)$$

as a weak perturbation for arbitrary V. In this case the overlap integral takes the form

$$\frac{(-1)}{2\pi^2} \int\!\!\int \frac{dE\, dE' (1 - n_{E'}) n_E}{(E - E' + io)^2} \mathrm{Tr}\{G_{\mathrm{on}}^R(E) \Delta V G_{\mathrm{on}}^R(E') \Delta V\}. \qquad (9)$$

Here n_ω is the Fermi distribution function, and G_{on}^R is the retarded on-shell Green function. Note that in the general case Green functions are matrices not only in the momentum space, but in the spin and electron-hole channels as well. Equation (9) can easily be rewritten in the form corresponding to Eq. (4). Thus we can write the expression for the function $f(\omega)$ [see Eq. (7)] as

$$\mathrm{Im} f_\omega = n p_F \int\!\!\int dE\, dE'\, \sigma(E, E') (n_{E'} - n_E) \times \delta(E - E' - \omega),$$

$$\sigma(E, E') = -\frac{1}{n p_F \pi R^2} \mathrm{Tr}\{G_{\mathrm{on}}^R(E) \Delta V G_{\mathrm{on}}^R(E') \Delta V\}, \qquad (10)$$

where $n = p_F^3/3\pi^2$ is the particle density of ^{3}He. The static limit of Eq. (10) gives the mobility in the elastic model $e/\mu = n p_F \int dE\, \sigma(E, E)(-dn_E/dE)$ (see Ref. [17]).

Until now we have not specified the superfluid phase of ^{3}He, and Eqs. (10) are valid for both ^{3}He-*A* and ^{3}He-*B*. To observe the ballistic motion of ions experimentally we need the scattering time to be very long ($\tau \gtrsim 1$–$10\ \mu$s). This condition under any reasonable experimental arrangements may be satisfied only in ^{3}He-*B* at $T \ll T_c$ [in fact, $T \leq (0.3$–$0.2) T_c$]. The R^2 expansion is valid while the recoil energy is smaller than the particle thermal energy. From the above discussion and the explicit calculation below it follows that the recoil energy with the renormalized mass is of order $(p_F^2/M)\Delta/\Gamma \sim \Delta/g$ which means that the R^2 expansion (and elastic scattering model) is valid for $T > T_{\mathrm{el}} = \Delta/g$. For large g (which for the electron bubble is between 10 and 30 depending on pressure) this temperature is quite low. An important point is that the effective mass renormalization is stopped at temperatures right below T_c, and at $T \ll T_c$ corrections to the effective mass due to normal excitations are exponentially small. In the region between T_c and T_{el} the particle starts moving ballistically with the exponentially increasing mean free path. Below T_{el} the ion recoil becomes important in calculating the diffusion rate, and the R^2 approximation breaks down. However, at these temperatures the mean free path is already many orders of magnitude larger than the particle wavelength, and the effective mass ceases to depend on T at all. Thus the calculation of M^{eff} within the R^2 expansion is fully justified, and in the rest of this Letter I concentrate on calculating M^{eff} from Eq. (10) in ^{3}He-*B* at $T = 0$.

It follows from the form of the effective action (2) that the mass renormalization is defined by the ω_n^2 term in the small frequency expansion of the functional integral

$$\delta M = -\frac{2}{\pi} \int_0^\infty \frac{d\omega\, \mathrm{Im} f(\omega)}{\omega^3}. \qquad (11)$$

(in ^{3}He-*B* the effective mass is isotropic). For $T = 0$ it can be further simplified to

$$\delta M = \frac{2 n p_F R_-^2}{\Delta} \int\!\!\int_0^\infty dx\, dx' \frac{\overline{\sigma}(x, -x')}{(x + x')^3}, \qquad (12)$$

where $\overline{\sigma}(x, x') = \sigma(x, x')/\pi R_-^2$ depends only on $(p_F R_-)$, and the dimensionless frequencies $x = E/\Delta$. In the simplest case of weak scattering potential (the strong-coupling limit $g \gg 1$ still may be realized through a large number of weak scattering channels contributing to σ) one can substitute the Green function in Eq. (10) by its unperturbed value, which is equivalent to performing a u-v Bogoliubov transformation on the normal state amplitude. After straightforward algebra we find

$$\delta M_{u-v} = \frac{\pi}{16}(1 + \sigma_2/2\sigma_{\rm tr})\frac{np_F\sigma_{\rm tr}}{\Delta}, \qquad (13)$$

where $\sigma_2 = \int d\Omega\,(1 - \cos^2\theta)d\sigma/d\Omega$. This mass renormalization could be as large as $4 \times 10^4 m_3$ at zero pressure. However, we demonstrate below that the exact calculation for the hard sphere potential gives a value of δM substantially different from Eq. (13).

As pointed out in Ref. [17] the scattering matrix has a resonant behavior at energies near the gap edge which has to be treated exactly. First, we express conventionally the Green function in terms of the scattering T matrix as $G(\omega) = V^{-1}T(\omega)V^{-1} - V^{-1}$ and present the trace in Eq. (10) in the form

$$\mathrm{Tr}\{T_{\rm on}^R(x)\Delta V^{-1}T_{\rm on}^R(x')\Delta V^{-1}\}, \qquad (14)$$

where $(\Delta V^{-1})_{\mathbf{pp}'} \equiv i(\mathbf{p} - \mathbf{p}')\mathbf{R}(V^{-1})_{\mathbf{pp}'}$. The analytic solution for the T matrix was found in Ref. [17]. Since the energy spectrum of ^{3}He-B is spherically symmetric and does not depend on spin, and $\boldsymbol{\sigma}\cdot\mathbf{p}$ (where $\boldsymbol{\sigma}$ is the fermion spin operator) is invariant under simultaneous rotations of the spin and momentum, it is clear that the T matrix is diagonal in the total angular momentum j and its projection m. Introducing the angular momentum eigenstates $|j,m,l = j \pm 1/2\rangle \equiv |jm\pm\rangle$ one can represent the T matrix as [17]

$$\pi N(0)T = \mathcal{T} = \begin{pmatrix} t_1(K) & -\sigma_2 t_3(-K)\sigma_2 \\ t_3(K) & \sigma_2 t_1(-K)\sigma_2 \end{pmatrix}; \qquad (15)$$

$$t_1 = \sum_{jm}\sum_{s=\pm}|s\rangle\langle s|t_1^{js};$$

$$t_3 = \sigma_2\sum_{jm}\sum_{s=\pm}|-s\rangle\langle s|t_3^{js},$$

where $N(0)$ is the density of states in the normal phase, and t_1^{js} and t_3^{js} are known functions of frequency and phase shifts at the Fermi surface, $K_{j\pm} = \tan\delta_{l=j\pm1/2}$

$$t_1^{js} = K_{js}(1 - i\rho K_{j-s})/d_{js};$$
$$t_3^{js} = (\rho/x)K_{js}K_{j-s}/d_{js}; \qquad (16)$$
$$d_{js} = (1 + i\rho K_{js})(1 - i\rho K_{j-s}) - (\rho/x)^2 K_{js}K_{j-s}.$$

Here $\rho(x) = x/(x^2 - 1)^{1/2}$. Obviously, we have the same matrix structure for the on-shell retarded T matrix as that in Eq. (15) with the scattering amplitudes being replaced by the on-shell ones $t \to \tau$

$$\tau_1^{js} = -i\theta(|x| - 1)\,\mathrm{Im}(t_1^{js});$$
$$\tau_3^{js} = -\theta(|x| - 1)\,\mathrm{Re}(t_3^{js}). \qquad (17)$$

It is easy to show that the nonzero contribution to the trace comes from terms having the scattering potential only in combinations $1/V_l - 1/V_{l'} = \pi N(0)(1/K_l - 1/K_{l'})$. So, we may conveniently replace the scattering potential in Eq. (14) with the K matrix. The final expression for $\sigma(x,x')$ reads

$$\sigma(x,x') = \frac{(-1)}{np_F\pi R^2}\mathrm{Tr}\{\mathcal{T}_{\rm on}^R(x)\Delta K^{-1}\mathcal{T}_{\rm on}^R(x')\Delta K^{-1}\}. \qquad (18)$$

There is one extra contribution to the effective mass due to bound states in the gap at $E_{js} = \pm\Delta\cos(\delta_{j+} - \delta_{j-})$ [17]. It is proportional to the number of occupied bound states multiplied by the ^{3}He quasiparticle mass

$$\delta M_b = m_3^* \sum_j^{j_{\rm max}}(2j + 1). \qquad (19)$$

The energies of the bound states approach the gap edge as $\Delta - E_j \sim \Delta(d\delta_j/dj)^2/2$. It is physically clear that states very close to the gap edge will disappear with the ion recoil in the scattering processes being taken into account. Thus the maximum orbital number contributing to Eq. (19) is defined by $\Delta - E_j \sim \Delta/g$. For the hard sphere potential with $p_F R_- \gg 1$ the phase shifts drop abruptly for $j > p_F R_-$, which allows us to estimate the contribution of the bound states as

$$\delta M_b \cong (p_F R_-)^2 m_3^* \approx 3\pi g m_3^*. \qquad (20)$$

This mass is likely to be larger than the bare ion mass M, but still is much smaller than the renormalization defined by virtual transition in Eq. (12).

The procedure of evaluating $M^{\rm eff}$ is straightforward now, because the trace determining the function $\sigma(E,E')$ can be expressed entirely in terms of phase shifts at the Fermi surface which for the hard sphere potential are defined as $\tan\delta_l = j_l(p_F R_-)/n_l(p_F R_-)$, where j_l and n_l are the spherical Bessel and Neumann functions of order l. For any given pressure the set of parameters $p_F(P)$, $\Delta(P)$, and $R_-(P)$ allows us to get the effective mass $M^{\rm eff} = M + \delta M_b + \delta M$ by numerical evaluation of Eqs. (12) and (18). In our calculations we used the normal state parameters taken from Wheatley's tabulation [18]. The ion radius was tabulated in Ref. [17]. Unfortunately, we found no tabulation for $\Delta(T \to 0, P)$ in ^{3}He-B, and had to rely on a weak-coupling relation $\Delta(P) = \alpha \times 1.76T_c(P)$ with the pressure-independent coefficient $\alpha = 1.12$ [8]. We think that $\Delta(P)$ is the most uncertain parameter in the present calculation. The bare ion mass is also unknown, but it is unlikely to contribute more than 10% to the effective mass, and we simply neglected this contribution.

Figure 1 shows the effective mass of the negative ion as a function of pressure. It was found to be as large as $\sim 2 \times 10^4 m_3$ at zero pressure dropping down to $(3\text{–}4) \times 10^3 m_3$ for $P > 10$ bars. The perturbative result, Eq. (13), is shown by the dashed line demonstrating the difference between the exact scattering amplitudes in ^{3}He-B and those obtained by applying the u-v transformation on the normal state amplitudes.

The prediction of a very large effective mass should clearly be tested experimentally. There is already circumstantial evidence for a large mass in the mobility experiments of Nummila, Simola, and Korhonen [8], who

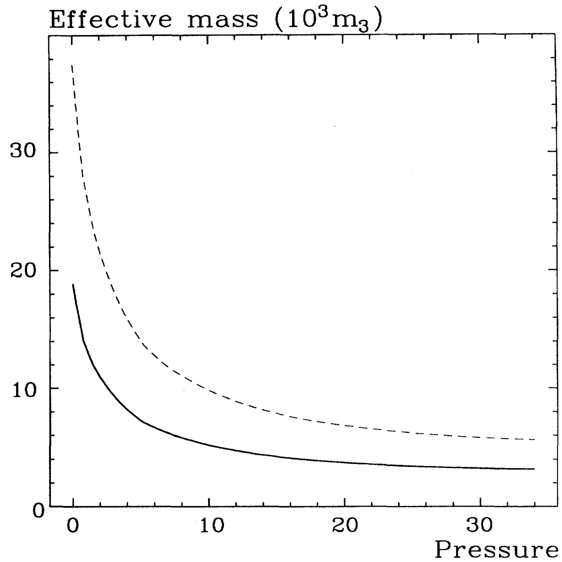

FIG. 1. Effective mass of the negative ion in ^{3}He-B. The solid line is the exact calculation for the hard-sphere potential which is compared to the perturbative result (dashed line) with the same transport cross section in the normal state.

found that the elastic scattering theory seemed to explain their data down to the lowest temperature obtained in ^{3}He-B; this would be hard to understand using a bare mass assumption, since it would give a recoil energy $\gg T$ already at $T = 0.4T_c$. However, the recoil energy calculated using the renormalized mass is much smaller (see above).

However, what we really require is a *direct* experimental test. One could search for resonant transitions between the ion energy levels near the liquid-vapor interface [20]. The distance between the ion and the surface is large as compared to the coherence length in ^{3}He-B up to the electric field strength $E \sim 100$ V/cm, with a typical range of resonance frequencies around $\omega_0 \sim 10$–40 MHz. From the mobility experiment [8] we estimate that $\omega_0\tau \gg 1$ below $0.3T_c$, and the resonance is sharp enough to be observed. In the time-of-flight experiment the ion mass in the bulk can be measured for arbitrary pressure. In this technique the electric field is reversed during a time interval $\Delta t \sim 1-10$ μs, and the ion is supposed not to accelerate above the Landau critical velocity $eE\Delta t/M^{\text{eff}} \leq \Delta/p_F$. Fortunately, the scattering time is as long as 1 μs already at $0.3T_c$, and the enormous ion mass effectively compensates the smallness of the critical velocity in ^{3}He-B.

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