"Intermittency" in Hydrodynamic Turbulence as Intermediate Asymptotics to Kolmogorov Scaling

Victor S. L'vov and Itamar Procaccia
Departments of Physics of Complex Systems and Chemical Physics, Weizmann Institute of Science, Rehovot 76100, Israel
(Received 28 February 1994)

A physical interpretation of a recent Navier-Stokes based theory for scaling in developed hydrodynamic turbulence is presented. It is proposed that corrections to the normal Kolmogorov scaling behavior of the nth order velocity structure functions are finite Reynolds number effects which disappear when the inertial interval exceeds 5–6 decades. These corrections originate from the correlation between the velocity differences and energy dissipation which are characterized by an anomalous (subcritical) exponent. The values of the experimentally observed scaling indices for the nth order structure functions for n between 4 and 14 are in agreement with our findings.

PACS numbers: 47.10.+g, 47.27.Gs

The desire to find a universal description of turbulence that was initiated by Richardson [1] in the 1920s, seemed for a while to be satisfied by the bold suggestion of Kolmogorov [2] in 1941 (K41) that one may construct a theory with one universal scaling exponent. This exponent was ascribed to differences of the longitudinal velocity across a scale R, $\delta u_\ell(\mathbf{x}+\mathbf{R},\mathbf{x}) \equiv [\mathbf{u}(\mathbf{x}+\mathbf{R}) - \mathbf{u}(\mathbf{x})] \cdot \mathbf{R}/R \sim R^{1/3}$, in the sense that the structure functions $S_n(R)$ of δu_ℓ satisfy the scaling laws

$$S_n(R) \equiv \langle[\delta u_\ell(\mathbf{x}+\mathbf{R},\mathbf{x})]^n\rangle \sim (\bar{\varepsilon}R)^{n\zeta_n} \sim (\bar{\varepsilon}R)^{n/3} \quad (1)$$

for values of R in the "inertial range" $L \gg R \gg \eta$ where L and η are the integral scale and the Kolmogorov dissipation scale, respectively. In (1) $\langle\cdots\rangle$ denotes an average over time, and $\bar{\varepsilon}$ is the mean of the dissipation field $\varepsilon(\mathbf{x},t) \equiv \nu[\partial_\alpha u_\beta(\mathbf{x},t) + \partial_\beta u_\alpha(\mathbf{x},t)]^2/2$ (ν is the kinematic viscosity). This suggestion of Kolmogorov was immediately attacked by Landau. Indeed, it is rather astonishing that a problem like fluid turbulence, which suffers from very large fluctuations and strong correlations, should be amenable to such a simple description; even Kolmogorov himself revised his thinking [3] and changed (1) to a more complicated form (which fell under attack as well). One measurement that raised a lot of objections to the K41 approach is the measurement of the correlation function of the dissipation field $K_{\varepsilon\varepsilon}(R)$

$$K_{\varepsilon\varepsilon}(R) = \langle\hat{\varepsilon}(\mathbf{x}+\mathbf{R},t)\bar{\varepsilon}(\mathbf{x},t)\rangle \propto R^{-\mu}, \quad (2)$$

where $\hat{\varepsilon}(\mathbf{x},t) = \varepsilon(\mathbf{x},t) - \bar{\varepsilon}$. It was found that $K_{\varepsilon\varepsilon}(R)$ decays very slowly in the inertial range, with μ being in the range 0.25–0.3 [4]. It was claimed [3,5–7] that the K41 theory required μ to vanish. Accordingly, there have been many attempts to construct models [3,5–7] of turbulence to take (2) into account and to explain how the measured deviations ($\zeta_n - 1/3$) in the exponents of the structure functions are related to μ.

In this Letter we make use of a recent description due to L'vov and Lebedev [8,9] of the mechanism for anomalous behavior of the energy dissipation field. In their theory μ is an independent nonzero scaling exponent while the K41 scaling is exact in the limit $\mathrm{Re} \longrightarrow \infty$. We shall argue, in contradiction with the common wisdom, that in the case $\mu = 0$ we expect a breakdown of K41 scaling behavior. For positive values of μ the K41 scaling of $S_n(R)$ should be valid in the limit of infinite Reynolds numbers Re. However, the observed value of μ is sufficiently small to be close to the borderline of the breakdown of K41. As a result there are essential corrections to (1) even at the largest experimentally available values of Re which are 10^8-10^9. We shall estimate these corrections and show that the known observations can be rationalized on the basis of a new picture of turbulence proposed here.

Our starting point is the Navier-Stokes equations, $\partial\mathbf{u}/\partial t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\nabla^2\mathbf{u} + \nabla p = 0, \nabla\cdot\mathbf{u} = 0$. These can be used to derive equations for $S_n(R,t)$, $\partial S_n(R,t)/\partial t + D_n(R,t) = J_n(R,t)$, which in the stationary state yields the balance, $D_n(R) = J_n(R)$,

$$D_n(R) \equiv n\langle[\delta u_\ell(\mathbf{x},\mathbf{x}')]^{n-1}\{\mathbf{P}_\ell[\mathbf{u}(\mathbf{x})\cdot\nabla\mathbf{u}(\mathbf{x})] - \mathbf{P}_\ell[\mathbf{u}(\mathbf{x}')\cdot\nabla\mathbf{u}(\mathbf{x}')]\}\rangle, \quad (3)$$

$$J_n(R) = \nu n\langle[\delta u_\ell(\mathbf{x},\mathbf{x}')]^{n-1}[\nabla^2 u_\ell(\mathbf{x}) - \nabla'^2 u_\ell(\mathbf{x}')]\rangle, \quad (4)$$

and $R = |\mathbf{x}-\mathbf{x}'|$, $\mathbf{P}_\ell = \{\mathbf{R}/R - [(\mathbf{R}/R)\cdot\nabla^{-2}\nabla]\nabla\}$.

In this Letter, rather than using formal diagrammatic expansions [9], we follow L'vov and Lebedev [8] in developing physical reasoning to expose the physical mechanism for anomalous scaling behavior. We begin with $K_{\varepsilon\varepsilon}(R)$. The reason for the existence of dissipation correlations between two points separated by an inertial-range distance R must be due to velocity components of size R and smaller. Eddies larger than R only sweep the two points $\mathbf{x},\mathbf{x}'$ together. Consider the simplest possible picture in which the velocity field $\mathbf{u}(\mathbf{x},t)$ contains only one typical scale R. One may understand this field as the result of filtering out (from a turbulent field with K41 spectrum) all the Fourier components lying in k space out of some shell of radius $k = 2\pi/R$. We denote such a field as $\mathbf{V}_R(\mathbf{x},t)$, and we can estimate its gradient

 0031-9007/95/74(14)/2690(4)$06.00

as $\mathbf{V}_R/R \sim (\bar{\varepsilon})^{1/3}R^{-2/3}$. In terms of $S_2(R)$ the situation may be exemplified as shown in Fig. 1(a). We have a bell-shaped contribution centered around $\ln R$, which in logarithmic scale has a width of W. Considering one R band without interband interactions is similar to applying Gaussian statistics, for which Eq. (2) yields $K_{\varepsilon\varepsilon}(R) \sim \nu^2(\bar{\varepsilon})^{4/3}R^{-8/3}$. Consider next a *two-band* flow in which there are two scales of eddies, R and r, with $R \gg r \gg \eta$. The structure function has now two bell-shaped contributions which in a scale invariant flow have the same logarithmic width W. In evaluating $K_{\varepsilon\varepsilon}(R)$ we shall interpret the ensemble average as a two-step process. The first one, $\langle\cdots\rangle_{r,R}$ is a conditional average on an ensemble of r eddies at a given velocity field $\mathbf{V}_R(\mathbf{x},t)$. The second, $\langle\cdots\rangle_R$, is an average over an ensemble of R eddies,

$$\langle\hat{\varepsilon}(\mathbf{x}_1,t)\hat{\varepsilon}(\mathbf{x}_2,t)\rangle = \langle\langle\hat{\varepsilon}(\mathbf{x}_1,t)\bar{\varepsilon}(\mathbf{x}_2,t)\rangle_{r,R}\rangle_R$$
$$= \langle\langle\hat{\varepsilon}(\mathbf{x}_1,t)\rangle_{r,R}\langle\hat{\varepsilon}(\mathbf{x}_2,t)\rangle_{r,R}\rangle_R\,. \quad (5)$$

The last step is justified because the dissipative field is mostly sensitive to the fast, small scale motion; each r-ensemble average can be done in the presence of some realization $\mathbf{V}_R(\mathbf{x},t)$ of the R eddies, and only when we compute the correlations we need to average in the R ensemble. Consider now the conditional average over small eddies $\langle\varepsilon(\mathbf{x}_1,t)\rangle_{r,R}$. It has a contribution from $|\nabla\mathbf{V}_R(\mathbf{x}_1,t)|^2$ which is almost not affected by the averaging over the small scales and a contribution from the small eddies denoted as $|\nabla\mathbf{V}_r(\mathbf{x}_1,t)|^2$ where $\mathbf{V}_r(\mathbf{x},t)$ is the velocity of r-scale eddies. Next order contributions come from the effect of $\nabla\mathbf{V}_R$ on the distribution function of ε fluctuations. We can therefore expand $\langle\varepsilon(\mathbf{x}_1,t)\rangle_{r,R}/\nu \approx |\nabla\mathbf{V}_r|^2 + |\nabla\mathbf{V}_R|^2 + a_{\alpha\beta}\partial_\alpha(V_R)_\beta + b|\nabla\mathbf{V}_R|^2 + c|\nabla\mathbf{V}_R|^3 + \cdots$ where we have displayed the tensor indices in the second term and suppressed them in the rest. The coefficients in the expansion, which in principle are functions of the scale r, are computed in the r ensemble which is isotropic and homogeneous. Therefore $a_{\alpha\beta} \propto \delta_{\alpha\beta}$ and the second term vanishes by incompressibility. It is clear that the coefficient $b(r)$ is dimensionless, and that $c(r)$ has the dimension of time. Since the functions $b(r)$ and $c(r)$ are characteristic of the r ensemble only, we can estimate them in terms of $\bar{\varepsilon}$ and r. Thus $b(r) \sim (\bar{\varepsilon}r)^0, c(r) \sim \bar{\varepsilon}^{1/3}r^{-2/3}$. In other words, b is a constant, independent of the scale r, whereas $c(r)$ is of the order of the turnover time τ_r. The ratio of the fourth to the third terms is a ratio of two time scales, $c|\nabla\mathbf{V}_R| \sim \tau_r/\tau_R$ which is small if $r \ll R$. Since $\langle\bar{\varepsilon}(\mathbf{x}_1,t)\rangle_{r,R} = \langle\varepsilon(\mathbf{x}_1,t)\rangle_{r,R} - \langle\langle\varepsilon\rangle_{r,R}\rangle_R$, we can write finally

$$\langle\hat{\varepsilon}(\mathbf{x}_1,t)\rangle_{r,R} \approx \nu|\nabla\widehat{\mathbf{V}_R(\mathbf{x}_1},t)|^2(1+b)\,, \quad (6)$$

where $|\nabla\widehat{\mathbf{V}_R(\mathbf{x}_1},t)|^2 = |\nabla\mathbf{V}_R(\mathbf{x}_1,t)|^2 - \langle|\nabla\mathbf{V}_R(\mathbf{x}_1,t)|^2\rangle_R$. The central point of the argument is that b is independent of the scale r. It implies that the same contribution $\nu b|\widehat{\nabla\mathbf{V}_R}|^2$ appears in any $\tilde{r}$ band for $\eta < \tilde{r} < R$. Upon summing up the contributions from the full spectrum [see Fig. 1(b)], we need to multiply one-band contribution by the number of statistically independent bands. This number is estimated as $(\ln R - \ln\eta)/W$ and

$$\sum_r\langle\hat{\varepsilon}(\mathbf{x}_1,t)\rangle_{r,R} \approx \nu|\nabla\widehat{\mathbf{V}_R(\mathbf{x}_1},t)|^2[1+\Delta\ln(R/\eta)]\,, \quad (7)$$

with $\Delta = b/W$. Notice that the width W is a measure of a characteristic length in logarithmic scales of the statistical dependence of the turbulent motion. Next consider the contribution of three groups of motions on scales R, x, and y, with $R \gg x \gg y \gg \eta$. We discussed already the contribution of the direct interaction between the scales R and x and R and y. Now we wish to evaluate the indirect two-step effect on the dissipation field due to effects of the largest R motion on the smallest y motions via the intermediate x motions. By repeating the above arguments we find instead of (6) $\langle\langle\hat{\varepsilon}(\mathbf{x}_1,t)\rangle_y\rangle_x \approx \nu|\widehat{\nabla\mathbf{V}_R}|^2(1+b_x+b_y+b_xb_y+\cdots)$ where we skip for brevity the additional indices in $\langle\langle\cdots\rangle_{y,x}\rangle_{x,R}$ which denote the conditions of averaging. Subscripts on the b's remind us of their origin. In fact, they are all the same, independent of scale. The number of contributions proportional to b^2 would have been $[(\ln R - \ln\eta)/W]^2$ if the relation between x and y were arbitrary. Since they are ordered in size $x < y$, we have only a half of that number. Finally

$$\langle\langle\bar{\varepsilon}(\mathbf{x}_1,t)\rangle_y\rangle_x \approx \nu|\widehat{\nabla\mathbf{V}_R}|^2(1+\Delta\ln(R/\eta)+\tfrac{1}{2}[\Delta\ln(R/\eta)]^2)\,.$$

Going on to n-step interactions we will find additional contributions $[\Delta\ln(R/\eta)]^n/n!$, where the $n!$ arises again due to the scale ordering $x > y > z > \cdots$. Such a series resums to

$$\sum_{\eta<z<y<x\cdots}\langle\langle\langle\hat{\varepsilon}(\mathbf{x}_1,t)\rangle_\eta\rangle_y\cdots\rangle \approx \nu|\nabla\widehat{\mathbf{V}_R(\mathbf{x}_1},t)|^2(R/\eta)^\Delta\,. \quad (8)$$

Using this result in (5) for both points $\mathbf{x}_1$, $\mathbf{x}_2$ and performing the last average over R motions we find

$$K_{\varepsilon\varepsilon}(R) \sim (\bar{\varepsilon})^{4/3}R^{-8/3}(R/\eta)^{2\Delta}\,. \quad (9)$$

This result was obtained in [8–10] on the basis of resummed diagrammatic expansions. Together with (2), Eq. (9) implies $\Delta = 4/3 - \mu/2 \approx 1.2$.

Armed with this understanding we can return now to analyze the balance equation, following the approach of

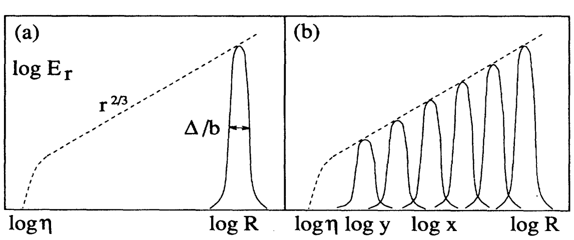

FIG. 1. Different scale eddy contributions to the structure functions (see text for details).

[10,11]. We start with J_n, Eq. (4). Adding $\mathbf{r}$ to $\mathbf{x}$ and $\mathbf{x}'$, replacing ∇_x and $\nabla_{x'}$ with ∇_r, and integrating once by parts we find

$$J_n(R) = -4\nu n(n-1)\langle[\delta u_\ell(\mathbf{x},\mathbf{x}')]^{n-2}\Xi(\mathbf{x},\mathbf{x}')\rangle, \quad (10)$$
$$(\mathbf{x},\mathbf{x}') \equiv [|\nabla u_\ell(\mathbf{x})|^2 - \nabla u_\ell(\mathbf{x}) \cdot \nabla u_\ell(\mathbf{x}')].$$

Consider first the "decoupled" part of $J_n(R)$, $\langle[\delta u_\ell)]^{n-2}\rangle \times \langle\Xi\rangle$. In isotropic flow $\nu\langle\Xi(\mathbf{x},\mathbf{x}')\rangle \approx \bar{\varepsilon}/3 - \nu S_2(R)/R^2$. The second term is negligible and

$$J_n^{\rm dec}(R) = -\tfrac{4}{3}n(n-1)\bar{\varepsilon}S_{n-2}(R). \quad (11)$$

Next, in the "coupled" contribution to (10) the term $|\nabla u_\ell(\mathbf{x})|^2$ contributes $\varepsilon(\mathbf{x})/3$, and the term $\nabla u_\ell(\mathbf{x}) \cdot \nabla' u_\ell(\mathbf{x}')$ gives a negligible contribution. The reason follows from the previous arguments. Repeating the multistep averaging process over the motions of intermediate scales from η to R, we get in each step a partly isotropized contribution for the next step. This multistep process will enhance the spherically symmetric part of the quantities averaged upon. The quantity $|\nabla u_\ell(\mathbf{x})|^2$, which is the gradient of the longitudinal part, gives, after averaging over angles, just a third of $\varepsilon(\mathbf{x})$. In contrast, a $\nabla u_\ell(\mathbf{x})$ term, as a vector, will contribute very little. Therefore

$$J_n^{\rm c}(R) \approx -\tfrac{4}{3}n(n-1)\langle[\delta u_\ell(\mathbf{x},\mathbf{x}')]^{n-2}[\varepsilon(\mathbf{x}) - \bar{\varepsilon}]\rangle$$
$$\approx \nu n(n-1)[C_n^{\rm red}S_n^{\rm red}(R) + C_n^{\rm irr}S_n^{\rm irr}(R)]\frac{R^{\Delta-2}}{\eta^\Delta}. \quad (12)$$

The last line follows from (8) and from estimating $|\nabla \mathbf{V}_R|^2$ as $\delta u_\ell(\mathbf{x},\mathbf{x}')^2/R^2$. In (12) $C_n^{\rm red}$ and $C_n^{\rm irr}$ are dimensionless coefficients, $C_2^{\rm red} = C_2^{\rm irr} = 0$. $S_n^{\rm red}(R)$ is the sum of all contributions to $S_n(R)$ which are obtained from decoupling the average $\langle \delta u_\ell^n \rangle$ into factors like $\langle \delta u_\ell^m \rangle \langle \delta u_\ell^{n-m} \rangle$, excluding the contributions taken in $J_n^{\rm dec}$. $S_n^{\rm irr}(R)$ is the remaining part that cannot be decoupled. Note that "*coupled*" and "*decoupled*" contributions are also "*reducible*" and "*irreducible*," but in a special sense, i.e., with respect to $|\nabla \mathbf{V}_R|^2$ which originated from ε.

To compare $J_n^{\rm c}$ with $J_n^{\rm dec}$ we can evaluate them using the K41 estimate $S_n(R) \sim (\bar{\varepsilon}R)^{n/3}$. The ratio $J_n^{\rm c}/J_n^{\rm dec} \sim (\eta/R)^{4/3-\Delta} = (\eta/R)^{\mu/2}$. We thus see that when Δ reaches a critical value of $\frac{4}{3}$, or equivalently when the exponent μ becomes zero, $J_n^{\rm c}$ is of the same importance as $J_n^{\rm dec}$. Since experimentally μ is not zero, but quite small, $\mu \approx 0.3$, we expect that $J_n^{\rm c}$ will introduce via the balance equation a visible correction to the scaling exponents even for very large Re. We shall estimate these corrections next.

We need first to estimate $D_n(R)$. As we discussed before, one gradient in a correlation is not likely to build up a multistep telescopic contribution. It can be substantiated that $D_n(R)$ can be estimated order by order in perturbation theory by local integrals [10,12]. We divide $D_n(R)$ in (3) into coupled and decoupled contributions in the same sense as in $J_n(R)$, i.e., with respect to δu^{n-2}, on the one hand, and the rest, on the other hand. There are $n - 1$ possibilities to do that. The resulting estimate is

$$D_n^{\rm dec}(r) = \tfrac{1}{2}n(n-1)D_2(R)S_{n-2}(R), \quad (13)$$

$$D_n^{\rm c}(r) \approx n[d_n^{\rm red}S_{n+1}^{\rm red}(R) + d_n^{\rm irr}S_{n+1}^{\rm irr}(R)]/R. \quad (14)$$

Here $d_n^{\rm red}$ and $d_n^{\rm irr}$ are dimensionless coefficients, $d_2^{\rm red} = 0$. For $n = 2$ Eqs. (11)–(14) give $D_2(R) \sim S_3(R)/R$, $J_2(R) \approx -8\bar{\varepsilon}/3$. The balance equation yields $S_3(R)/R \sim \bar{\varepsilon}$. We find no correction to $S_3(R)$, as is expected by the requirement that $S_3(R) = -\frac{4}{5}\varepsilon R + 6\nu dS_2(R)/dR$. Note that (14) is valid only for $n \geq 2$, so we do not generate corrections to S_2 in our approach either. In order to find corrections for $n \geq 4$ consider the balance equation $D_n(R) = J_n(R)$ for $n \geq 3$. Equations (11) and (13) show that the decoupled contributions on both sides are identical. Thus $D_n^{\rm c}(R) = J_n^{\rm c}(R)$ which together with (12) and (14) gives

$$[d_n^{\rm red}S_{n+1}^{\rm red}(R) + d_n^{\rm irr}S_{n+1}^{\rm irr}(R)]$$
$$\approx (\bar{\varepsilon}R)^{1/3}(\eta/R)^{\mu/2}[C_n^{\rm red}S_n^{\rm red}(R) + C_n^{\rm irr}S_n^{\rm irr}(R)]. \quad (15)$$

The right-hand side of (15) vanishes in the limit $\eta/R \to 0$. Therefore $S_{n+1}^{\rm irr} \sim S_{n+1}^{\rm red} \propto (\bar{\varepsilon}R)^{n/3}$. This means that in the limit Re $\longrightarrow \infty$ we recover the K41 scaling of the structure functions. For nonzero η/R one may consider (15) as a recurrence relation which expresses $S_{n+1}^{\rm irr}$ in terms of lower order structure functions S_m, $m \leq n$. Considering in (15) the case $n = 3$ and using the fact that $S_3(R) \sim \bar{\varepsilon}R$ we find the first nontrivial correction in $S_4^{\rm irr}$, namely $\sim \nu S_3(R)R^{\Delta-2}/\mu^\Delta$. Finally

$$S_4(R) = S_4^{\rm red} + S_4^{\rm irr} \sim (\bar{\varepsilon}R)^{4/3}\left[1 + \tilde{C}_4\left(\frac{\eta}{R}\right)^{\mu/2}\right], \quad (16)$$

with $\tilde{C}_4 \simeq C_3^{\rm irr}/d_4^{\rm red}$. We stress that the value $\mu = 0$ corresponds to nondecaying correlations of the dissipation as a function R. In that case the physics of turbulence should change completely destroying K41 even in the limit Re $\longrightarrow \infty$. There is an indication of such a possibility in the problem of turbulent diffusion of a passive scalar field [10]. In Navier-Stokes turbulence, with $\mu > 0$, the correction term vanishes when $\eta/R \longrightarrow 0$. Notwithstanding, due to the smallness of μ, for any appreciable value of $\tilde{C}_4$ we need an enormously large inertial range before $\tilde{C}_4(\eta/R)^{\mu/2}$ becomes negligible compared to unity. The experimental evidence [13,14] is that the coefficients involved are larger than 7. For $\mu \simeq 0.3$ the correction term is actually *dominant* as long as $\eta/R > 10^{-5}$. At the present values of experimentally available Re there is no case in which there exist 5 orders of magnitude of inertial range. It would thus appear that $S_4(R)$ scales nicely with an apparent exponent whose value is $4/3 - \mu/2$.

The comparison of our theoretical considerations with experiments becomes more difficult for $n > 4$, since the data analysis in all experiments to date did not take into account the contributions of the reducible parts to S_n. The experimental data analysis can be correct only if the irreducible parts of S_n are always much larger than the reducible ones. If we adopt this assumption, the recurrence relation (15) would result in the prediction

$$n\zeta_n = n/3 - \mu(n - 3)/2. \qquad (17)$$

This prediction can be compared with experiments. In recent experiment [15] the values of $n\zeta_n$ shown in Table I were reported. Choosing $\mu = 0.276$ we get the theoretical values shown in the table. For comparison the K41 values are also displayed. We stress, however, that the excellent agreement between theory and experiment cannot be taken too seriously due to the unavoidable existence of reducible contributions. It just shows that there are no glaring contradictions.

Equation (17) has a superficial similarity to the β model, in which $n\zeta_n = n/3 - \mu(n - 3)/3$. One major difference is that the normalizing length scale in (16) is η and not L. This is an experimentally verifiable difference that has not been properly tested yet. A second major difference is that in the β model this correction is believed to be asymptotic also at Re $\longrightarrow \infty$ while (17) is an intermediate asymptotic result for large but finite Re. Also, our theory does not exclude the possibility that the existence of non-negligible contributions with K41 scaling may influence the apparent scaling exponents in a way that results in a nonlinear dependence $n\zeta_n$ vs n. These differences reflect completely different physics. The theoretical message is that the K41 scaling for the structure functions remains exact in the limit Re $\longrightarrow \infty$, and probably leaves non-negligible contributions at experimentally relevant values of Re. We propose that the experimental data analysis should be redone in the light of this theory to subtract the reducible part of S_n before performing log-log plots. This should separate the contributions with K41 exponents from those with anomalous exponents. In doing so one has to be also aware of other possible corrections to K41 which stem from noncritical mechanisms, like the corrections $\delta\zeta_2 \sim (\eta/L)^{2/5}$ which are due to the anisotropy of the excitation of turbulence on the outer scale L [16,17]. Such corrections may lead to differences in the measured exponents in different experiments.

In summary, we have presented a new theory of scaling behavior in turbulence at finite Re. In contrast to various phenomenological models of intermittency our approach is based on the Navier-Stokes equations. The first cornerstone in this theory is the result [18] that the K41 scaling of velocity differences is exact in the limit Re $\longrightarrow \infty$. The second cornerstone is the theoretical understanding of how fields that are sensitive to the dissipative scale, like the energy dissipation field, exhibit anomalous scaling with nontrivial exponents [8–10]. The last cornerstone is the idea that the dissipation field is subcritical. Its scaling exponent $\mu/2$ is very small: $\mu/2 \simeq 0.1-0.2$. We have shown here that the Navier-Stokes equations impose a constraint in the form of the balance equation which feeds back the anomaly of the dissipative field onto the scaling of the structure functions. Since the exponent μ is positive, the feedback effect disappears at infinite Re. Nevertheless, due to the smallness of $\mu/2$ the subcritical corrections to scaling remain important for all experimentally realizable Re. It is noteworthy that our theory explains why the observed deviations from K41 are so small: There exists a small parameter in the theory, i.e., $\mu/2$, which allows us to find small corrections to K41. This leaves the fascinating theoretical problem of understanding whether the small value of μ is accidental, or whether it stems from fundamental reasons.

Critical comments by K.R. Sreenivasan were very instructive for us. This work has been supported in part by the Minerva Foundation, Munich, Germany and the Basic Research Fund of the Israeli Academy.

TABLE I. Comparison between experimental and theoretical values of the scaling exponents. Experiment [15]: turbulent boundary layer at (based on its thickness) Re = 32 000.

n	2	4	6	8	10	12	14
$n\zeta_n$ (expt.)	0.70	1.20	1.62	2.00	2.36	2.68	3.02
$n\zeta_n$ (theor.)	—	1.20	1.59	1.98	2.37	2.76	3.15
$n\zeta_n$ (K41)	0.67	1.33	2.00	2.67	3.33	4.00	4.67

[1] L. Richardson, Proc. R. Soc. London A **110**, 709 (1926).
[2] A.N. Kolmogorov, Dokl. Akad. Nauk. USSR **38**, 538 (1941).
[3] A.N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).
[4] K.R. Sreenivasan and P. Kailasnath, Phys. Fluids A **5**, 512 (1993).
[5] E. Novikov and R. Stewart, Izv. Geoph. **3**, 408 (1964).
[6] B.B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).
[7] U. Frisch, P. Sulem, and M. Nelkin, J. Fluid Mech. **87**, 719 (1978).
[8] V.S. L'vov and V.V. Lebedev, Europhys. Lett. (to be published).
[9] V. V. Lebedev and V.S. L'vov, JETP Lett. **59**, 577 (1994).
[10] V.S. L'vov, I. Procaccia, and A.L. Fairhall, Phys. Rev. E **50**, 4684 (1994).
[11] R.H. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
[12] V.S. L'vov and I. Procaccia (to be published).
[13] A. Praskovsky and S. Oncley, Europhys. Lett. **28**, 635 (1994).
[14] V.S. L'vov and I. Procaccia, Europhys. Lett. **29**, 291 (1995).
[15] G. Stolovitzky, K.R. Sreenivasan, and A. Juneja, Phys. Rev. E **48**, R3217 (1993).
[16] V.S. L'vov and I. Procaccia, Phys. Rev. E **49**, 4044 (1994).
[17] S. Grossmann, D. Lohse, V.S. L'vov, and I. Procaccia, Phys. Rev. Lett. **73**, 432 (1994).
[18] V.S. L'vov, Phys. Rep. **207**, 1 (1991).