Quantized Energy Cascade and Log-Poisson Statistics in Fully Developed Turbulence

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It is proposed that the statistics of the inertial range of fully developed turbulence can be described by a *quantized* random multiplicative process. We then show that (i) the cascade process must be a log-infinitely divisible stochastic process (i.e., stationary independent log-increments); (ii) the inertial-range statistics of turbulent fluctuations, such as the coarse-grained energy dissipation, are log-Poisson; and (iii) a recently proposed scaling model [Z.-S. She and E. Leveque **72**, 336 (1994)] of fully developed turbulence can be derived. A general theory using the Levy-Khinchine representation for infinitely divisible cascade processes is presented, which allows for a classification of scaling behaviors of various strongly nonlinear dissipative systems.

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The statistics of fully developed turbulence exhibit certain universal features as a result of strong nonlinear interactions. One set of intriguing quantities characterizing universal behavior of turbulent flows is a set of scaling exponents for two-point correlation functions, e.g., ζ_p of the velocity structure functions defined by an expression $\langle \delta v_\ell^p \rangle \sim \ell^{\zeta_p}$, where $\delta v_\ell = v(x+\ell) - v(x)$ is a (longitudinal) velocity difference across a distance ℓ. A similar set of quantities is an exponent τ_p for pth-order moment of locally averaged energy dissipation ϵ over a ball of size ℓ: $\langle \epsilon_\ell^p \rangle \sim \ell^{\tau_p}$. The range of the length scale ℓ for the above power-law behavior to be valid is called an inertial range. Kolmogorov's refined similarity hypothesis [1] provides a relation between these two sets of quantities: $\zeta_p = p/3 + \tau_{p/3}$. This Letter addresses a predictive model [2] of ζ_p and τ_p. Note that ζ_2 characterizes the scaling for the kinetic energy fluctuations and is directly related to the exponent for the kinetic energy spectrum $E(k) \sim k^{-\alpha}$: $\alpha = 1 + \zeta_2$.

During the past half century since Kolmogorov's 1941 seminal work [3] that predicts $\tau_p = 0$ and $\zeta_p = p/3$, there has been a continual effort to experimentally determine the values of ζ_p or τ_p. There is now strong evidence [4–7] that $\zeta_p \neq p/3$, which is usually referred to as the intermittency effects or anomalous scaling exponents. Many theoretical models have been proposed to address this phenomenon. Some approaches start with an ansatz of probability density function (PDF) of ϵ_ℓ such as the log-normal model [2] or a log-stable model [8], for example. Others [9–12] propose discrete random multiplicative processes (RMP) modeling the energy cascade. The stochastic process that generates the energy cascade is furnished by random multiplicative factors $\mathcal{W}_{\ell_1\ell_2}$ relating the fluctuations of ϵ_ℓ at two different length scales ℓ_1 and ℓ_2: $\epsilon_{\ell_2} = \mathcal{W}_{\ell_1\ell_2}\epsilon_{\ell_1}$. By construction, the multiplicative factor is independent of ϵ_{ℓ_1}. The probability distribution of $\mathcal{W}_{\ell_1\ell_2}$ determines τ_p, since it is required that $\log\langle \mathcal{W}_{\ell_1\ell_2}^p \rangle / \log(\ell_2/\ell_1) = \tau_p$. Existing cascade models of random multiplicative processes [10–12] are obtained by making an *ad hoc* ansatz for the PDF of $\mathcal{W}_{\ell_1\ell_2}$ being composed of a certain number of discrete atoms described by one or more adjustable parameters. The parameters in the models are difficult to determine by plausible physical arguments. It is important to note that $\mathcal{W}_{\ell_1\ell_2}$ specifies an *underlying* process; its realization may not be the instantaneous ratio $\epsilon_{\ell_2}/\epsilon_{\ell_1}$. In other words, the dynamical meaning of RMP is yet unclear. Models based on log-normal or log-stable generators have additional difficulties at large p; they violate certain exact inequalities [13,14] (see also [9]).

Recently, She and Leveque (SL) [1] proposed a new scaling model, hereafter referred to as the SL model, based on a characterization of turbulence in terms of a hierarchy of eddies with excitation levels defined by successive moments: $\epsilon_\ell^{(p)} = \langle \epsilon_\ell^{p+1} \rangle / \langle \epsilon_\ell^p \rangle$. Each group of eddies $\epsilon_\ell^{(p)}$ have a specific scaling behavior. The most singular group $\epsilon_\ell^{(\infty)}$ is assumed to be filamentary structures observed in both laboratory experiments [15] and numerical simulations [16,17]; their scaling exponent was derived based on physical considerations [1]. Scalings of less singular eddies $\epsilon_\ell^{(p)}$ ($p < \infty$) are related to that of the most singular eddies $\epsilon_\ell^{(\infty)}$ by a recursive relation

$$\epsilon_\ell^{(p+1)} \sim \epsilon_\ell^{(p)\beta} \epsilon_\ell^{(\infty)1-\beta}. \quad (1)$$

In such a setting, the scaling exponents τ_p are completely determined by the characteristics of the most singular structures, i.e., their scaling exponent and the geometrical dimension of their support. A prediction is made: $\tau_p = -2p/3 + 2[1 - (2/3)^p]$. The resulting scaling exponents for the velocity structure function

$$\zeta_p = p/9 + 2[1 - (2/3)^{p/3}] \quad (2)$$

is in striking agreement with the most recent experimental values obtained by Benzi *et al.* [5] for a fully developed turbulent flow behind a cylinder (see Table I).

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TABLE I. Comparison of the experimentally measured scaling exponents ζ_p [5] and the theoretical prediction [Eq. (2)] from the SL scaling model.

Order p	Experiment [5] ζ_p	SL model [1] ζ_p	K41 [3] $\zeta_p = p/3$
2	0.37	0.364	0.333
2	0.70	0.696	0.667
3	1.00	1.000	1.000
4	1.28	1.279	1.333
5	1.54	1.538	1.667
6	1.78	1.778	2.000
7	2.00	2.001	2.333
8	2.23	2.211	2.667

The most dramatic assumption (and the most innovative idea) in the SL paper [1] is the recursive scaling relation (1) between different groups of eddies. It was speculated [1] that Eq. (1) underlies some hidden symmetry in the solution of the Navier-Stokes equations. Here, we propose a quantized random cascade picture over a continuum of scales, in which either an event leading to the formation of a characteristic singular structure occurs or such an event modulated by a defect. The defect adds a finite amount of disorder to the singular-structure event. The quantization of a cascade means that any cascade event can be represented by the singular-structure event modulated by an integer number of defects. It will be shown that this quantized random cascade leads to a log-Poisson statistics for inertial-range turbulent fluctuations, from which the SL scaling result and Eq. (1) follow. Furthermore, we discuss the consequences of a valid random multiplicative description over a continuum of scales for the inertial-range dynamics of the Navier-Stokes system. It is shown that the statistics of the inertial-range fluctuations must be log-infinitely divisible. With a general decomposition theorem of infinitely divisible processes, we establish the quantized log-Poisson process described above as the simplest scenario (one singular structure and one quantum) of cascade consistent with the infinite divisibility. This analysis allows for a discussion of universality classes of more general strongly nonlinear dissipative dynamics.

Assume that the inertial-range dynamics *underlies* a random multiplicative process; the content of this assumption will be discussed in detail later. We can then define the cascade multiplicative factor $\mathcal{W}_{\ell_1\ell_2}$ for any arbitrary pair of length scales ($\ell_1\ell_2$, $\ell_1 > \ell_2$):

$$\epsilon_{\ell_2} = \mathcal{W}_{\ell_1\ell_2}\epsilon_{\ell_1}, \qquad \ln\langle \mathcal{W}^p_{\ell_1\ell_2}\rangle = \tau_p(\ln\ell_2 - \ln\ell_1) \quad (3)$$

(this does *not* imply that $\mathcal{W}_{\ell_1\ell_2} = \epsilon_{\ell_2}/\epsilon_{\ell_1}$). Consider an infinitesimal change of the length scale $\delta = \ln\ell_1 - \ln\ell_2 \to 0^+$. Assume $\mathcal{W}_{\ell_1\ell_2} = \mathcal{W}_\delta$ has two events. The first takes a value $\alpha_\delta = (\ell_1/\ell_2)^\gamma = \exp(\gamma\delta) > 1$, and $\gamma = O(1)$, characterizing the rate of the amplification of the dissipation event ϵ_{ℓ_1}, is assumed positive. This amplification leads to a singularity with a rate of divergence $\alpha_\delta = \exp(\gamma\delta)$. We call this event a singular structure event. The second event takes a value $\alpha_\delta\beta$ with $\beta < 1$ and corresponds to a modulation of the singular structure event by the factor β. This event will be called a modulation-defect event. In the limit $\delta \to 0$, $\alpha_\delta \to 1$; but $\beta < 1$, meaning the modulation is performed in a discrete (quantized) way. In order to make $\langle \mathcal{W}_{\ell_1\ell_2}\rangle = 1$, the probability to observe a modulation defect must go to zero in proportion to δ. It is then simple to show that over a finite separation of scales $\ell_1\ell_n$, where $n\delta = O(1)$, the probability to observe $X = m$ modulation defects obeys a Poisson law,

$$P(X = m) = \exp(-\lambda_{\ell_1\ell_2})\frac{\lambda^m_{\ell_1\ell_2}}{m!},$$

where $\lambda_{\ell_1\ell_2} \propto \ln\ell_1 - \ln\ell_2$ is the mean. Over the range $\ell_1\ell_n$, the events of m-modulation defects correspond to a multiplication factor $\mathcal{W}_{\ell_1\ell_2} = \beta^m\alpha^n_\delta = \beta^m\exp\gamma$; therefore, $\mathcal{W}_{\ell_1\ell_2}$ has a log-Poisson distribution. The quantized structure here is that $\ln\mathcal{W}_{\ell_1\ell_2}$ always contains an integer number of quanta: $m\ln\beta$.

The effect of a modulation defect is to reduce the fluctuation of the dissipation, when going from large to small scales. It is observed from numerical studies of the Navier-Stokes turbulence [18] that small amplitude fluctuation events are very disordered and random, while high amplitude fluctuation events show remarkable coherence. The above cascade picture suggests that small-scale disordered eddies are generated from large-scale structures via a quantized (discrete) process which involves multiple occurrence of modulation defects. Each modulation defect makes a finite contribution to the destruction of the coherence of an ideal singular structure.

We now show that this simple log-Poisson process for $\mathcal{W}_{\ell_1\ell_2}$ gives rise to the SL scaling laws [1]. Let $\mathcal{W}_{\ell_1\ell_2} = (\ell_1/\ell_2)^\gamma\beta^X$, where γ, β are constants, and X is a Poisson random variable with mean $\lambda_{\ell_1\ell_2}$. A straightforward calculation shows that

$$\ln E\mathcal{W}^p_{\ell_1\ell_2} = p\ln\alpha_{\ell_1\ell_2} + \lambda_{\ell_1\ell_2}(\beta^p - 1). \quad (4)$$

Taking into account the continuity constraint $E\mathcal{W}^1_{\ell_1\ell_2} = 1$, we get $\lambda_{\ell_1\ell_2} = -\gamma(\ln\ell_1/\ell_2)/(\beta - 1)$, and obtain

$$\ln E\mathcal{W}^p_{\ell_1\ell_2} = \left(p - \frac{\beta^p - 1}{\beta - 1}\right)\ln\alpha_{\ell_1\ell_2}. \quad (5)$$

Taking $\gamma = 2/3$ and $\beta = 2/3$, we obtain the result of She and Leveque [1]. It can be further shown that

$$\mathcal{W}^{(p+1)}_{\ell_1\ell_2} = \mathcal{W}^{(p)\beta}_{\ell_1\ell_2}\alpha^{1-\beta},$$

where $\mathcal{W}^{(p)}_{\ell_1\ell_2}$ is defined similarly as $\epsilon^{(p)}_\ell$ and $\mathcal{W}^{(\infty)}_{\ell_1\ell_2} = \alpha$. This establishes the validity of Eq. (1).

In the SL scaling model [1], the parameters γ and β can be explicitly related to the properties of the most singular structures in turbulence, and can thus be estimated based on physical consideration. The rate of the divergence of the most singular structure is γ which is related to β and D_∞, the "dimension" of the level set $\epsilon_\ell^{(\infty)}$; see [1] for the definitions of level sets and the parameter D_∞. From Eq. (5) and the definition of γ, we obtain

$$\tau_p = -\gamma p + (d - D_\infty)(1 - \beta^p), \qquad (6)$$

with $\beta = 1 - \gamma/(d - D_\infty)$; here d is the dimension of the space. It is clear that $0 \leq \beta \leq 1$; it follows that $0 \leq \gamma \leq d - D_\infty$. For $\beta = 2/3$, we have an interesting relation: $\gamma = (d - D_\infty)/3$. Recently, Benzi [19] shows that $\beta = 2/3$ also gives an approximate description of the scaling laws in the shell model of fully developed turbulence, with a different D_∞. It suggests that D_∞ as a characterization of the most singular structures is system dependent, but β may be constant in a wider universality class; see more discussions below.

Over a finite range of scales, a quantized cascade provides a log-Poisson process containing an infinite number of discrete values for the multipliers, or in probabilistic terms, an infinite number of atoms. This is in strong contrast to previously proposed discrete cascade models which contain a finite number of atoms. For example, the α model [9,11] and p model [12] contain two atoms, and the random β model [10] contains three atoms. The log-Poisson model realizes a very important property of the random multiplicative process, the log-infinite divisibility, which none of the models with a finite number of finite atoms possess. The β model [20] is log-infinitely divisible because it has an infinite atom at $\ln \mathcal{W}_{\ell_1 \ell_2} = -\infty$. In general, in view of the log-infinite divisibility requirement, any cascade models based on a finitely many finite atoms cannot represent such a continuous scaling process as the inertial range of turbulence.

The assumption of the existence of a random multiplicative process for the inertial-range dynamics is key to the present analysis and needs more discussion here. We conjecture that it is achieved because of the strongly nonlinear coupling. An intuitive argument goes as follows: The statistical states of fluctuations at inertial-range scales are linked exclusively by nonlinear couplings through which all (phase) information propagates. At the stationary state, mixing of the phase information is abundant, because a large number of dynamic degrees of freedom participate in the dynamics, rendering extremely chaotic motions. Hence, the multiplicative links between large and small eddies may become statistically independent of the property of the particular eddies concerned, and the independence holds irrespective of the relative size of the two eddies. A necessary condition for its existence is obtained from Eqs. (3), which formally define $\mathcal{W}_{\ell_1 \ell_2}$ through the inverse Laplace transform. Because the left-hand side (l.h.s.) of (3) is a convex function and $\ln(\ell_2/\ell_1) < 0$, τ_p must be a concave function of p. Experimentally observed τ_p is indeed a concave function of p. The concavity of τ_p rules out, however, a similar random multiplicative description in the opposite direction for, e.g., $\epsilon_{\ell_1} = \mathcal{W}_{\ell_2 \ell_1} \epsilon_{\ell_2}$. As we switch ℓ_1 and ℓ_2, $\ln(\ell_2/\ell_1)$ changes sign, and the r.h.s. of (3) changes from a convex to a concave function, and this is in contradiction with the convexity of the l.h.s. of (3). This suggests that, if given a cascade interpretation of the multiplicative process, the concavity of τ_p prescribes the direction of the cascade; see Gupta and Waymire [21] for a further discussion in another context. Physically, the concavity of τ_p is a result of the breaking of the time-reversal symmetry in the Navier-Stokes system. Indeed, the inviscid (Euler) equation is invariant under the time reversal transformation ($t \to -t$, $\mathbf{u} \to -\mathbf{u}$), which reverses the direction of the mean energy flux. Only the viscous dissipation breaks this symmetry, so that the forward energy cascade dynamics (to small scales) are not governed by the same equation as the backward energy cascade, which, we believe, is the origin of the concavity of τ_p. Further theoretical study on this point will be of interest to pursue.

If a random multiplicative process prevails, then there is a random multiplicative factor for any pair of length scales (ℓ_1, ℓ_2), since turbulent inertial-range dynamics is a continuous scaling process. The usual notion of cascade "steps" should be regarded merely as a convenient way to describe a process which must eventually be taken to some continuous limit to arrive at the description over a continuum of scales. It follows that one can insert an arbitrary number of length scales between ℓ_1 and ℓ_2, and write $\mathcal{W}_{\ell_1 \ell_2}$ as a product of many $\mathcal{W}$'s which are all defined similarly by a relation like (3). In other words, $\ln \mathcal{W}_{\ell_1 \ell_2}$ can be written as a sum of many independent $\ln \mathcal{W}$'s. Furthermore, by the definition (3), the statistical distribution of $\mathcal{W}_{\ell_1 \ell_2}$ depends on scales ℓ_1, ℓ_2 only through the ratio ℓ_2/ℓ_1. This is a translational symmetry $\ln \ell \to \ln \ell + \ln \lambda$, implying that all of the inserted $\ln \mathcal{W}$ factors have the same distribution. Therefore, $\ln \mathcal{W}_{\ell_1 \ell_2}$ must be an infinitely divisible random variable, and the log-generator process is a process with stationary independent increments (in $\ln \ell$), a property for infinitely divisible processes, see Bhattacharya and Waymire [22].

The distribution of an infinitely divisible process is completely determined by a possible continuous (i.e., no jumps) Gaussian (Brownian motion) contribution with dispersion parameter σ^2, and by a limiting sum of independent compound Poisson processes and an attendent Poisson parameter and a Levy-Khinchine measure K governing the average size of jumps; see Bhattacharya and Waymire [22] for an explanation. This provides a representation of τ_p of the following form (when it exists):

$$\tau_p = -mp + \frac{\sigma^2}{2} p^2 + \int_{-\infty}^{\infty} \left[e^{-yp} - 1 + \frac{py}{1 + y^2} \right] K(dy).$$

Included in this representation are the stable processes with exponent $0 < s \leq 2$ and the gamma process, for example, with an appropriate measure $K(dy)$. With the exception of the Gaussian case, the moment generating function, hence τ_p, is not convergent for all values of p for these examples. This may be used as a basis to rule out such examples. The quadratic term is due to the Gaussian part of the representation, but for $\sigma^2 > 0$ the Novikov inequalities [13] are violated at large enough p. Because of this inequality, the log-stable distributions such as those considered in [8] with $s \approx 1.65$ can also be ruled out.

If we assume that the jump measure K is atomic, and contains a single atom y_0, then we immediately obtain the log-Poisson model described above with $K(dy) = 2\delta_{-\ln\beta}(dy)$, ($y_0 = -\ln\beta$). Thus, from the viewpoint of the above general theory, the SL scaling law corresponds to a case in which there is only one singular structure ($\gamma = 2/3$ for filaments) and one quanta ($\beta = 2/3$). This distinguishes the 3D incompressible Navier-Stokes turbulence from other nonlinear dissipative systems. It may be possible in some systems to have multiple singular structures, with multiple modulation defects associated with their own dynamics: $(\gamma^{(1)}, \beta^{(1)})$, $(\gamma^{(2)}, \beta^{(2)})$, etc. Then, a compound Poisson statistic will arise, instead of a simple Poisson process. This raises an issue about the universality classes of general nonlinear dissipative systems. A proper definition of the universality class may be the number of singular structure and their associated quanta (modulation defect). Then, systems even having different scaling exponents (such as the shell model with different hyperviscosities [23], or the wall-bounded turbulence at different distance from the wall, or anisotropic turbulence with various degrees of anisotropy at large scales) may still belong to the same universality class. On the other hand, systems such as compressible turbulence may contain multiple structures (filaments and sheets) and multiple quanta (associated with the dynamics of both the solenoidal part and compressible part) and may belong to a different universality class. The existence of the variety of possible scenarios for the scaling behavior of nonlinear dissipative systems, as seen from the Levy-Khinchine representation, is one of the interesting and concrete results of the present work, which will require both further theoretical investigations and experimental verifications. The mathematical foundations largely concern discrete scale cascades, e.g., [24,25], although Kahane's T-martingale theory also permits certain notions of continuous scale. Finally, we add that a deductive link between the probabilistic description provided here and the nonlinear PDE describing the dissipative dynamics is still lacking. Many interesting challenges are yet ahead.

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