Avalanches in Continuum Driven Dissipative Systems

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We show that continuum driven dissipative systems can have behavior analogous to that of cellular automata sandpile models. We discuss the conditions for this behavior, and argue that a system which can be shown to satisfy these criteria will exhibit a broad power law distribution of discrete energy dissipating avalanches. We give a one-dimensional example of such a system, and show that such behavior does result when these conditions are met. We then show that the statistical behavior of this system can be approximated by a much simpler cellular automaton. Finally, we argue that a general prescription for showing how certain real physical systems can be understood in this manner.

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Cellular automaton (CA) models of sandpiles can be naturally driven to a scale invariant state, a phenomenon dubbed "self-organized criticality" (SOC) [1]. This property has been invoked to explain the dynamics of driven dissipative systems which exhibit a broad power law distribution of discrete energy dissipating events, such as earthquakes and solar flares [2,3]. However, these cellular models are inherently discontinuous since they are solved on a discrete grid with the time updating done in a series of discontinuous steps. This poses the problem of how such CA models can quantitatively mimic the behavior of continuum systems, such as the coronal magnetic field of solar active regions or the stress along a geologic fault, which have no obvious cellular structure and whose evolution does not proceed in discrete steps. Carlson and Langer [4] have given an example of a temporally continuous model consisting of discrete blocks that exhibits a broad distribution of avalanche-type events. However, there has not previously been a demonstration that a continuum field model can exhibit SOC-type behavior. In this Letter we address the question of what physical properties of continuum driven dissipative systems lead to such avalanche behavior. We then give an example of such a system and show that under certain conditions it will display a scale invariant avalanche distribution. We then show how a CA model can reproduce the behavior of this system. This work provides a prescription for arguing that systems such as solar active regions can be described as SOC-type systems.

In a typical CA sandpile model, a field ϕ is defined on a spatial grid. Another field, which we will term the instability field $I(\phi)$, is a local functional of ϕ. The field is defined to be locally unstable at a particular position if $|I(\phi)|$ there exceeds some critical value. When a point becomes unstable, ϕ is readjusted by diffusing it among the neighboring points so as to reduce $|I(\phi)|$ there, and in the process making the original location now stable. This may force neighboring points to become unstable. The evolution of the system proceeds by incrementing ϕ at some location on the grid. The field is then simultaneously readjusted at any unstable points. If any new unstable locations have been created in this readjustment step, the field is then readjusted at these locations. This readjustment step is iterated until the field relaxes to a state which is stable everywhere. This event, from the onset of an instability to the eventual stabilization of the field, is termed an avalanche. The field is then incremented again at some location, and this process is continued. The field eventually reaches a state where the distribution of avalanches becomes stationary.

We use the properties of these CA models to guide us in understanding how a continuum system can have analogous behavior. First, the field of the continuum system must possess a local threshold instability which causes rapid local diffusion of the field. For the system to be physically reasonable, the local instability must eventually be self-stabilizing [i.e., locally reduce $|I(\phi)|$], and must dissipate energy in the field. In the CA model, the field configuration is stable, and no readjustment takes place as long as the instability criterion is not met. In addition, the system is only driven during the time between the occurrences of avalanches. In a continuum system, these conditions correspond to the limit in which the instability time scale is very much faster than both the evolutionary time scale when an instability is not occurring and the driving time scale. Thus whenever an instability is occurring, the effect of the driving term is negligible. In addition, the slow evolution that takes place when no instability is present must be much slower than the driving so that the structure in the field configuration does not dissipate away during the time period between avalanches. For the avalanches to be distinct, the driving time scale must also be slow enough so that on average not more than a single avalanche is occurring at the same time.

A crucial aspect of these CA models is that avalanches of all sizes occur, and that the distribution is scale invariant over a large range. If the avalanche process is governed by a local conservation law, then the only way in which field can be removed from a region is for it to be transported out through the boundary by the action of avalanches. Consider a subregion within the

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system of radius r. In steady state, the amount of field added to this region by the driving term must equal the amount of field removed from this region by the action of all avalanches. If avalanches are short ranged, and only reach out to a maximum length λ from their point of origin, then only those avalanches originating within distance λ of the boundary of the region can change the amount of field within the volume. The volume of the region contributing to the outward flux of field scales as $r^{d-1}\lambda$, where d is the dimensionality of space, while the total volume of the subregion scales as r^d. Thus the total amount of field escaping from the subregion cannot match the field added to the subregion as r becomes large. It is not possible for increased flux from the surface of the subregion to make up for this shortfall since the threshold nature of the instability does not allow arbitrarily large amounts of the instability field to build up in any location (since at the start and end of an avalanche, the field is in a stable configuration). Therefore, avalanches cannot have a characteristic decay length, and there must be large avalanches which cover regions approaching the system size. If any characteristic length scale in the dynamical equations is very much less than the size of the system, the fact that large avalanches must exist suggests that there will be a power law distribution of avalanches over intermediate length scales (as is observed). Furthermore, if the energy dissipated per unit volume in an instability is independent of the size of an avalanche, then the distribution of avalanches as a function of energy will also be a power law. We found [2] that if the redistribution law does not conserve the field (or the instability field), then the distribution of avalanches is not a power law. However, as we shall show, the existence of a conservation law alone does not guarantee a power law distribution of avalanches.

We note that, if instead of readjusting $|I(\phi)|$ to a value below the instability threshold, an instability reduces it to a value equal to the threshold, then the overall behavior of the system will be very different. Eventually the instability field will build up to a value which is everywhere equal to the threshold. The addition of field to any point will then result in an instability, and this additional bit of field will be transported to the boundary. All avalanches can only terminate at the boundary. In order for the system to have avalanches which can terminate within the region, there must be locations on the lattice which can remain stable even if a neighboring point becomes unstable [5]. This is possible if the redistribution of field in an instability lowers the magnitude of the instability field to a value below the threshold. Thus, the dynamical equations have a memory which allows an instability to carry the local field configuration past the point at which it becomes unstable to begin with. The system of differential equations describing such a system must be at least second order in time, since a first order in time evolution equation cannot have such behavior. Equivalently, the dynamical equations for the field can be written as n coupled first order equations, with $n \geq 2$. We note, e.g., that the first order system studied by Diaz-Guilera [6] will not exhibit a power law distribution of avalanches.

To summarize, we expect that driven dissipative systems which meet the following criteria can be naturally driven to a state with a broad power law distribution of discrete energy dissipating events. First, there must be a locally conserved field which is subject to a threshold instability. The instability leads to rapid local diffusion of the field, which dissipates energy and locally stabilizes the field. The field must have metastable states in which energy can be built up by the driving term. The slow evolutionary time scale of these metastable states must be very much slower than the rapid instability time scale, and the driving time scale must be intermediate between the two. Furthermore, any intrinsic length scales associated with the instability must be very much less than the size of the system. Finally, the system must be driven for a long enough period for it to settle to a steady state.

A simple one-dimensional example of a system which meets the above requirements is given by

$$\frac{\partial \phi(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[D(x,t)\frac{\partial \phi}{\partial x}\right] + S(x,t), \tag{1a}$$

$$\frac{\partial D(x,t)}{\partial t} = \frac{Q(|\partial \phi/\partial x|)}{\tau} - \frac{D(x,t)}{\tau}, \tag{1b}$$

$$Q(|\partial \phi/\partial x|) = \begin{cases} D_{\min}, & \text{low state, } \partial \phi/\partial x| < k\,, \\ D_{\max}, & \text{high state, } |\partial \phi/\partial x| > \beta k\,, \end{cases} \tag{1c}$$

where $\phi(x,t)$ is a scalar field, $D(x,t)$ is a spatially and temporally varying diffusion term, $S(x,t)$ is the source term, and $D_{\max} \gg D_{\min}$. The function Q is double valued and dependent upon the history for $\beta k < |\partial \phi/\partial x| < k$, where k is the instability threshold and $0 < \beta < 1$. The value of Q remains in the low state $Q = D_{\min}$ until the slope $|\partial \phi/\partial x| > k$, whereupon it undergoes a transition to the high state. However, when in the high state, $Q = D_{\max}$, it will not make the transition to the low state until the slope $|\partial \phi/\partial x| < \beta k$. Thus Q acts very much like a physical instability in that the value of the slope k required to turn on the instability is greater than the slope βk required to maintain the instability once it is turned on. When Q changes to $D_{\max}$, the diffusive term $D(x,t)$ begins to rise toward $D_{\max}$. If the local slope remains above k for a time longer than of order τ then the diffusion coefficient will saturate at $D = D_{\max}$. Once the instability is turned off, the diffusion coefficient decays back to $D_{\min}$ with time constant τ.

If the field is everywhere stable, the evolution of ϕ will be governed by the source term S for small $D_{\min}$. However, eventually at some location the slope will exceed the threshold, $|\partial \phi/\partial x| > k$. As D increases, the field ϕ then begins to diffuse at this location, which has the effect of locally reducing the slope $|\partial \phi/\partial x|$ while at the same time increasing the slope at nearby locations, so that the instability can spread. If $D_{\max}$ is sufficiently large, the diffusive term can dominate over the driving term S, and the effect of the driving term is negligible over the course of an avalanche. Eventually the field returns to a state which is everywhere stable, and D is again everywhere small. There are a number of intrinsic length and time scales in this system. The constant τ sets a characteristic time scale for an instability. The characteristic length $\delta x = \sqrt{\tau D_{\max}}$ is of order of the minimum size region over which $|\partial\phi/\partial x| > k$ where D can build up to value $D_{\max}$ before the instability turns off.

We define the energy density by the square of the field ϕ^2, similar to the field energy in a magnetic system, the stress energy in an elastic medium, or the gravitational potential energy of a pile of sand. The effect of an instability is to reduce the total energy in the field since $\partial/\partial t \int \phi^2(x,t)\,dx = -2\int D(x,t)(\partial\phi/\partial x)^2 dx < 0$, where we have used the boundary condition $\phi = 0$ at $x = 0, L$. Note that the energy dissipation rate per unit volume for avalanches of any size greater than δx is of order $D_{\max}k^2$. Thus, the total energy will fluctuate, slowly increasing due to the driving term (for $\langle S\rangle \neq 0$), and rapidly decreasing in bursts due to the diffusive term.

We solve Eq. (1) using an explicit finite difference code. We consider Eq. (1) as if it were a phenomenological equation of motion for a physical system. As with all dynamical equations describing macroscopic physical systems, Eq. (1) will be valid only down to some limiting length and time scale, below which the relevant physics is different. In that spirit, we consider the continuum limit of the discretized solutions to Eq. (1) where the grid size Δx and time step Δt represents the minimum length and time scale of validity of Eq. (1). Furthermore, we will consider the behavior of solutions of Eq. (1) over times much longer than τ, and for δx much smaller than the size of the system. We define the field $\phi(x,t)$ at each spatial grid point, while the diffusion term $D(x,t)$ and the slope $\partial\phi/\partial x$ are defined at the half-grid points. To check if we have numerically resolved the equations, we solved a number of initial value problems with $S = 0$, $D_{\max} = 5$, $D_{\min} = 0$, $\tau = 1$, and $\beta = 0.9$. We find that due to the discontinuous nature of Q, these numerical solutions develop significant structure in $d\phi/dx$ and D on length scales down to the grid spacing during the course of an avalanche. However, as Δx becomes very small compared to δx, we find that the evolution of $\phi(x,t)$ and the total field energy become approximately independent of the grid size and time step. This system is quite sensitive to initial conditions due to the threshold nature of Q, and the numerical solutions with decreasing Δx do not necessarily converge. While these solutions are not truly resolved, these differences are small, and the macroscopic evolution of the system for these initial value problems is nearly independent of the lower length cutoff. However, a small but nonzero $D_{\min}$ and the addition of a small diffusive term to Eq. (1b) make the solutions smooth and numerically resolvable without substantially changing the macroscopic behavior.

Next we investigate the long term behavior of the system for continuous driving. We solve Eq. (1) over the domain $0 < x < 40$, with the driving term $S(x,t) = S_0[20 - |20 - x|]$. The driving term increases the slope $|\partial\phi/\partial x|$ everywhere at the same rate (except, of course, at the midpoint $x = 20$). We choose $\Delta x = 0.1$, $D_{\max} = 5$, $D_{\min} = 0$, $\tau = 1$, $k = 0.04$, $\beta = 0.9$, and $S_0 = 3 \times 10^{-6}$, for which $\delta x = 2.2$ and the above criteria are satisfied. We start the field in a random configuration, and find that it eventually reaches a statistical steady state which is independent of the initial state. We find that the energy in the field fluctuates as in the CA models, steadily rising due to the driving term, but punctuated by rapid decreases due to avalanches. In the inset of Fig. 1 we show a typical time series of the energy dissipation rate, given by the negative of the time derivative of the total energy. In Fig. 1 we show a histogram of the distribution of avalanches as a function of energy dissipation. The distribution is approximately power law over a large range, $F(E) = F_0E^{-\zeta}$, with $\zeta \approx 1$. The precise value of ζ is somewhat uncertain due to the relatively small number of avalanches (a total of 353 events are tabulated). The lower cutoff to the power law $E_{\min}$ is of order of the energy dissipated in a region of size δx in time τ; $E_{\min} \approx \tau\delta x D_{\max}k^2 \approx 2 \times 10^{-2}$. We find that the slope ζ is not strongly sensitive to the driving rate S_0, the size of the domain, the threshold k, or β for $0.7 < \beta < 0.95$. The behavior of this system is therefore quite similar to that of the sandpile automata in that the macroscopic behavior is robust to changes in the microscopic equations, provided that the conditions given earlier remain valid.

Reducing the grid spacing Δx does not change the power law index ζ, but does lead to more frequent avalanches. This is due to the fact that there is structure on smaller

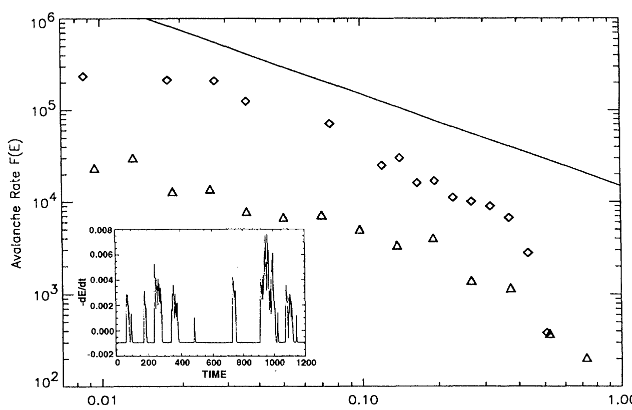

FIG. 1. Avalanche rate vs energy for the continuum system (triangles) and for the CA model (diamonds). The values of the parameters for the continuum system are such that the inequality conditions given in the text are satisfied. The curves have been shifted for clarity. For reference the solid line has logarithmic slope -1. Inset: energy dissipation rate vs time.

length scales, so that the likelihood of the slope at some point exceeding k increases. However, the total energy added to the system is roughly the same, and must equal the total energy dissipated by all avalanches, $\int EF(E)\,dE = [F_0/(2-\zeta)]E_{\max}^{-\zeta+2} \approx F_0 E_{\max}$. The fact that smaller grid spacing leads to an increase in F_0 requires a corresponding decrease in $E_{\max}$. For smaller and smaller Δx, the avalanches become more and more frequent until eventually the avalanches merge and are no longer distinct. For small but nonzero $D_{\min}$, we find that the overall avalanche rate F_0 eventually stabilizes as Δx is decreased. However, for the parameters listed above, as $D_{\min}$ is increased above $\sim 10^{-6}$, the behavior of the solutions changes greatly, with mostly large avalanches occurring. The distribution becomes a broad peak from energy approximately 0.3 to 1.5; in other words, all avalanches are system-wide events. This is because the period of time between avalanches is greatly increased since the diffusive term smooths out the large peaks in $\partial\phi/\partial x$, which are the trigger sites for avalanches. By the time an avalanche is triggered, the field has built up to an extent that the characteristic instability length is of order of the system size. This is an instance in which the avalanche distribution is not a power law, even though a conservation law for ϕ exists. A similar behavior (an excess of large events) occurs as $\beta \rightarrow 1$ because for small events Q rapidly reverts to the low state before D can rise to an appreciable value. This has the same effect as a nonzero $D_{\min}$. This behavior is in part due to the nature of the driving term which has constant slope. If instead we drive the system with $dS(x,t)/dx$ having spatial and temporal variations so that localized regions may have much higher driving rates, the avalanche rate is increased, and again a power law distribution of avalanches can result if the rate is large enough so that small regions can become unstable. In both these cases with non-power-law avalanche distributions, the slow evolutionary time scale is fast enough to allow significant dissipation of structures between avalanches. Note that an excess of large events is seen both in real sandpiles [7] and in the sliding block model of Carlson and Langer [4].

We next demonstrate that the statistical behavior of this system can be approximated by a much simpler CA model which by construction trivially satisfies the above constraints. We define the field ϕ at discrete grid points separated by the characteristic length $\delta x = 2.2$. The field is locally unstable whenever the slope between any two adjacent points exceeds k. When this occurs we decrease the field ϕ at the higher point while simultaneously increasing the value of ϕ by an equal amount at the lower point, so as to reduce the slope between them by k. We drive the system by adding a random small quantity $\delta\phi$ to the field ϕ at a random location. The evolution proceeds as for the CA model described earlier. While the driving does not strictly mimic that of Eq. (1) since the average field added is independent of position, we find that for $\delta\phi/\Delta x > 0.2k$ avalanches are initiated roughly independently of position, with a slightly larger number initiated at the edges $x = 0, L$. Avalanches are triggered by randomness in the driving term, rather than by fluctuations in the configuration of ϕ. In Fig. 1 we compare the avalanche distribution for the CA model with that of Eq. (1). The two distributions are quite similar, with the upper rollovers differing due to the difference in avalanche rate. We note that the CA avalanche distribution deviates somewhat from a power law due to the small number of grid points $L/\delta x = 18$. The continuum distribution suffers from this same problem, in addition to the statistical uncertainty mentioned earlier.

More importantly, the general overall behavior of the system is preserved. The CA model is far simpler than Eq. (1), yet it retains the essential physics which leads to the macroscopic avalanche behavior. An important advantage in making use of such CA models is the fact that the computational time may be orders of magnitude faster than solving the full equations of a much more complex system. Even for the relatively simple 1D system given by Eq. (1), several days of computational time is required to tabulate enough events to make Fig. 1.

For a system as large and complex as the magnetized coronal plasma of a solar active region, a full solution of the equations of motion is well beyond present capabilities. Even if it were possible, complete numerical solutions of the equations of motion would probably also not be very illuminating due to their overwhelming complexity. In cases such as this we argue that if the system can be shown to satisfy the criteria given above, then while the microphysics may not be fully understood, one can still hope to understand some of the macroscopic behavior [8]. Furthermore, this can provide a justification for the use of sandpile type CA models in modeling their evolution. An attractive feature of these models is their conceptual simplicity, which may provide some insight into the behavior of the full system.

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[1] P. Bak, C. Tang, and K. Weisenfeld, Phys. Rev. Lett. **59**, 381 (1987); L. Kadanoff, S. Nagel, L. Wu, and S. Zhou, Phys. Rev. A **39**, 6524 (1989).

[2] E.T. Lu and R.J. Hamilton, Astrophys. J. **380**, L89 (1991); E.T. Lu, R.J. Hamilton, J.M. McTiernan, and K. Bromund, *ibid.* **412**, 841 (1993).

[3] P. Bak, Physica (Amsterdam) **163A**, 403 (1990).

[4] J.M. Carlson and J.S. Langer, Phys. Rev. Lett. **62**, 2632 (1989).

[5] J. Carlson, J. Chayes, R. Grannan, and G. Swindle, Phys. Rev. A **42**, 2467 (1990).

[6] A. Diaz-Guilera, Europhys. Lett. **26**, 177 (1994).

[7] H. Jaeger, C. Liu, and S. Nagel, Phys. Rev. Lett. **62**, 40 (1989).

[8] E.T. Lu (to be published).