Late-Time Tail of Wave Propagation on Curved Spacetime

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The late-time behavior of waves propagating on a class of curved spacetimes, including Schwarzschild, is studied. The late-time tail is not necessarily an inverse power of time. Our work extends, places in context, and provides understanding for the known results for the Schwarzschild spacetime. Analytic and numerical results are in excellent agreement.

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Waves propagating on curved spacetime develop "tails." A pulse of gravitational waves (or other massless fields) travels not only along the light cone, but also spreads out behind it, and slowly dies off in tails [1–6]. This tail phenomenon is fascinating theoretically, and has also been found in post-Newtonian calculations to have possibly observable secular effects on the phase of the orbit of inspiralling binary systems [7].

For asymptotically late times this tail often has a particularly simple behavior, namely, it decays as an inverse power in t. Detailed analyses have been carried out for the Schwarzschild geometry, using both analytic [2,3,5,6] and numerical techniques [2,4]. These works are based on the Regge-Wheeler perturbation formulation, in which the propagation of linearized gravitational, electromagnetic, and scalar waves is described by the Klein-Gordon (KG) equation

$$D\phi(x,t) \equiv [\partial_t^2 - \partial_x^2 + V(x)]\phi(x,t) = 0, \qquad (1)$$

with $V(x)$ describing the scattering of ϕ by the background geometry. The late-time tail has been explained in two different ways: in terms of a branch cut in the Green's function in the frequency plane [6], or in terms of scattering from large radius in the Schwarzschild geometry [2,3].

With the power-law tails in Schwarzschild spacetime well established (although a thorough understanding, especially at timelike infinity, is not complete [8]), attention has been widened to more general situations. Tails in a Reissner-Nordstrom hole have been studied [3]. Moreover, since the late-time tail comes from scattering at large radius in the Schwarzschild case [2], it has been suggested that power-law tails would develop even when there is no horizon in the background [3], implying that such tails should be present in perturbations of stars, or after the collapse of a massless field which does not result in black hole formation. In [4], the late-time behavior of scalar waves evolving in its own gravitational field or in gravitational fields generated by other scalar field sources was studied numerically. Power-law tails have been reported for all these cases in numerical experiments, though with exponents different from the Schwarzschild case. These interesting results call for a systematic study of the late-time tail in general nonvacuum, time-dependent, and nonlinear spacetimes.

There are several interesting questions in a general analysis. Is the tail always a power in t? What determines the asymtotic behavior—the branch cut in the Green's function or the asymptotic form of the potential—and how are the two related? What determines the magnitude and the time dependence of the tail, and do these depend on local geometry and/or the presence of a horizon?

In this Letter, we study these questions using Eq. (1) with a broad class of $V(x)$. Our work extends, and places in context, the known results for linearized waves on specific time-independent background geometries which are asymptotically flat. More general cases, involving, for example, nonlinear waves and time-dependent backgrounds [4], would need to be dealt with separately.

We shall first present numerical simulations of the late-time behavior for various $V(x)$, showing that the decay is not necessarily an inverse power of t. Another interesting observation is that when the parameters of the potentials are continuously varied, the behavior of the late-time tail can change discontinuously. We then determine analytically the amplitude and the time dependence of the late-time tail in terms of the strength of the branch cut in the Green's function. The relation between the cut and the asymptotic structure of the potential is obtained. We establish that the local properties of the potential affect only the magnitude but not the time dependence of the late-time tail.

We study numerical evolutions of $\phi(x,t)$ given by Eq. (1) for various $V(x)$. The variable x is related to, but not the same as, the circumferential radius r [9]. For a nonsingular metric, e.g., that of a star, $r \in (0,\infty)$ maps into $x \in (0,\infty)$. For a metric with an event horizon at $r = r_0$ (with $g_{tt} = 0$ at r_0 [9]), then $r \in (r_0,\infty)$ typically maps into $x \in (-\infty,\infty)$ (the tortoise coordinate). The evolutions shown here are for the half-line $x \in (0,\infty)$ case, with boundary conditions $\phi = 0$ at $x = 0$ and outgoing waves for $x \to +\infty$. The full-line case [$x \in (-\infty,\infty)$, with outgoing wave boundary conditions for $|x| \to \infty$] is basically the same.

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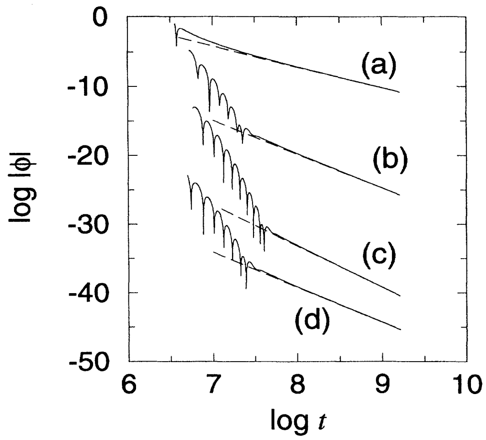

FIG. 1. $\ln|\phi|$ vs $\ln t$, for several power-law potentials. (a) $l = 0$, $\alpha = 3$; (b) $l = 1$, $\alpha = 2.9$; (c) $l = 1$, $\alpha = 3$; (d) $l = 1$, $\alpha = 3.1$. Solid lines are numerical evolutions from generic initial data while dashed lines are analytical results. They are indistinguishable for $\ln t > 8$. To make the four sets of lines stagger, vertical shifts have been applied: (b) downwards by 6.0; (c) downwards by 12.0; (d) downwards by 23.0.

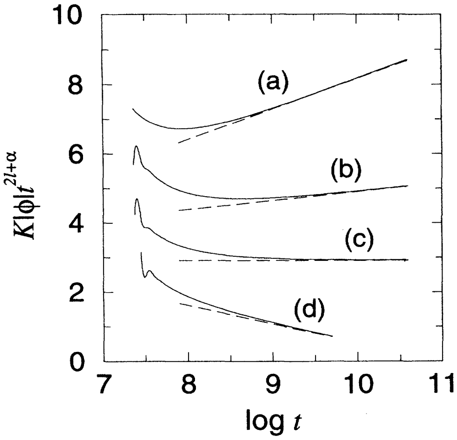

FIG. 2. $K|\phi|t^{2l+\alpha}$ vs $\ln t$ for several logarithmic potentials (a) $l = 0$, $\alpha = 3$; (b) $l = 1$, $\alpha = 2.9$; (c) $l = 1$, $\alpha = 3$; (d) $l = 1$, $\alpha = 3.1$. Solid lines are numerical results while dashed lines are analytical results. They are indistinguishable for $\ln t > 9.5$. For clarity, the data are multiplied by a constant K with (a) $K = 10^{-9}$; (b) and (c) $K = 5.6 \times 10^{-10}$; (d) $K = 5.86 \times 10^{-10}$.

In the following we consider two classes of $V(x)$: potentials which go as a centrifugal barrier $l(l+1)/x^2$ (l is an integer) plus $\overline{V}(x)$, with $\overline{V}(x)$ being (i) $x_0^{\alpha-2}/x^\alpha$ ("power-law potentials") or (ii) $(x_0^{\alpha-2}/x^\alpha)\ln(x/x_0)$ ("logarithmic potentials") when $x \to +\infty$ for some x_0. The logarithmic potential includes the Schwarzschild metric as a special case. The evolution is basically independent (see below) of the initial data. The cases reported here use Gaussian initial data for ϕ and $d\phi/dt$.

Figure 1 shows $\ln|\phi|$ vs $\ln t$ at some fixed point x for several power-law potentials. Solid lines represent the numerical evolutions; earlier times are suppressed for clarity. After some quasinormal mode ringing, they approach and coincide with the analytic results (to be derived below) representing power-law decays $t^{-\mu}$, where $\mu = 2l + \alpha$ except that in case (c) μ jumps discontinuously to $2l + 2\alpha - 2$. Such jumps occur whenever α is an odd integer less than $2l + 3$. (We assume throughout that the initial $d\phi/dt$ is not exactly zero; otherwise the exponents μ increase by 1, also shown below.)

Logarithmic potentials often lead to logarithmic late-time tails. To exhibit this behavior, Fig. 2 shows $|\phi|t^{2l+\alpha}$ vs $\ln t$ for several logarithmic potentials. The numerical evolutions approach and coincide with the analytic results which represent decays in the form of $t^{-\mu}(\ln t)^\beta$, with $\mu = 2l + \alpha$, and $\beta = 1$, except that β jumps discontinuously to 0 (and the late-time tail becomes a power law) in case (c); such jumps again occur when α goes through an odd integer less than $2l + 3$.

Table I summarizes these and other cases studied but not shown here. In all these examples, α is taken to be larger than 2 ($\alpha \le 2$ will be discussed elsewhere).

We first present a heuristic picture, and then state the necessary modifications. Consider a wave from a source point y reaching a distant observer at x. The late-time tail is caused by the wave first propagating to a point $x' \gg x$, being scattered by $V(x')$, and then returning to x, arriving at a time $t \simeq (x' - y) + (x' - x) \simeq 2x'$. Thus at late times $\phi \propto V(x') \simeq V(t/2)$. In particular, if $V(x) \sim x^{-\alpha}(\ln x)^\beta$, then one expects the late-time tail to be $\sim t^{-\alpha}(\ln t)^\beta$.

This picture requires two modifications. First, a centrifugal barrier, corresponding to free propagation in three dimensions, does not contribute to the late-time tail, so that it is the remainder of the potential $\overline{V}(x)$ that matters. For $\overline{V}(x) \sim x^{-\alpha}(\ln x)^\beta$, the late-time tail turns out to be $t^{-(2l+\alpha)}(\ln t)^\beta$. The suppression by a factor t^{-2l}, at least in the case $\alpha = 3$, is known for specific black hole geometries [2,3]. Second, if α is an odd integer less than $2l + 3$, the leading term in the late-time tail vanishes. For $\beta = 0$, the next leading term is expected to be $t^{-(2l+2\alpha-2)}$, while for $\beta = 1$, the next leading term is $t^{-(2l+\alpha)}$ without a $(\ln t)$ factor.

Next we present a full analytic treatment for the half-line problem; modifications for the full-line case are

TABLE I. Behavior of late-time tails for potentials going as $l(l+1)/x^2 + \overline{V}(x)$ when $x \to \infty$.

$\overline{V}(x), x \to \infty$	$\alpha > 2$	$\phi(t),\ t \to \infty$
$\frac{x_0^{\alpha-2}}{x^\alpha}$	Odd integer $<2l+3$	$t^{-\mu}, \mu > 2l + \alpha$
	All other real α	$t^{-(2l+\alpha)}$
$\frac{x_0^{\alpha-2}}{x^\alpha}\ln\left(\frac{x}{x_0}\right)$	Odd integer $<2l+3$	$t^{-(2l+\alpha)}$
	All other real α	$t^{-(2l+\alpha)}\ln t$

straightforward. The evolution of $\phi(x,t)$ described by Eq. (1) is

$$\phi(x,t) = \int dy[G\dot{\phi}(y,0) + \dot{G}\phi(y,0)] \tag{2}$$

for $t > 0$, where the retarded Green's function $G(x,y;t)$ is defined by $DG = \delta(t)\delta(x-y)$ with $G = 0$ for $t < 0$ and the outgoing wave boundary condition as given for ϕ above.

The Fourier transform $\tilde{G}$ satisfies

$$\tilde{D}(\omega)\tilde{G} \equiv [-\omega^2 - \partial_x^2 + V(x)]\tilde{G}(x,y;\omega) = \delta(x-y) \tag{3}$$

and is analytic in the upper half ω plane. Define auxiliary functions f and g by $\tilde{D}(\omega)f(\omega,x) = \tilde{D}(\omega)g(\omega,x) = 0$, where g satisfies $\lim_{x\to\infty}[e^{-i\omega x}g(\omega,x)] = 1$, and f satisfies $f(\omega, x=0) = 0$, $f'(\omega, x=0) = 1$ [10] for the half-line problem, and $\lim_{x\to-\infty}[e^{i\omega x}f(\omega,x)] = 1$ for the full-line problem. In terms of f and g, and henceforth assuming $y < x$, $\tilde{G}(x,y;\omega) = f(\omega,y)g(\omega,x)/W(\omega)$, where the Wronskian $W(\omega) = W(g,f) = g(df/dx) - f(dg/dx)$ is independent of x. Now express G in terms of $\tilde{G}$ and, for $t > 0$, distort the contour for the Fourier integral to a large semicircle in the lower half ω plane. One therefore identifies three contributions to G, as follows.

First, the integral on the large semicircle can be shown to vanish beyond a certain time $t_P(x,y) = O(x)$, and does not affect the late-time behavior [11].

Second, at the zeros of the Wronskian $W(\omega)$ at $\omega = \omega_j$, $j = \pm1, \pm2, \ldots$ on the lower half plane, f and g are proportional to each other, and each of them satisfies *both* the left and the right boundary conditions. Hence they are quasinormal modes (QNM's), and their collective contribution G_Q vanishes exponentially at late times. [The case where $G = G_Q$ (QNM's being complete) has been discussed in detail [11].]

Finally, there may be singularities of f and g in ω, which lead to the late-time tail. If the potential $V(x)$ has finite support, say on $(0,a)$, then one can impose the right boundary condition at $x = a^+$. Integrating through a *finite* distance with a nonsingular equation to obtain $g(\omega,x)$ cannot lead to any singularity in ω. It is not surprising that the same holds if $V(x)$ vanishes sufficiently rapidly as $x \to +\infty$ [11,12]. However, if $V(x)$ has an inverse-power type tail, then $g(\omega,x)$ will have singularities on the negative Imω axis (see below), in the form of a branch cut, as in the Schwarzschild case [6]. The cut extends to $\omega = 0$, and its tip controls the late-time behavior. For the half-line problem, $f(\omega,x)$ is integrated from $x = 0$ through a finite distance, and hence does not have any singularities in ω. For the full-line case, f is dealt with in the same manner as g. In all cases of interest, the tail of $V(x)$ as $x \to -\infty$ is either faster than any exponential or precisely exponential. For the former, $f(\omega,x)$ has no singularities in ω, while for the latter, there will be a series of poles, but at a finite distance from $\omega = 0$. In either case, the spatial asymptotics as $x \to -\infty$ has no bearing on the late-time behavior.

It then remains to study the spatial asymptotics as $x \to +\infty$, and the consequent singularities of g. First consider a power-law potential with $l = 0$. Applying the first Born approximation to $\tilde{D}(\omega)g = 0$ and starting with $e^{i\omega x}$ as the zeroth order solution, it is readily shown that

$$g(\omega,x) \simeq e^{i\omega x} - I(\omega,x), \tag{4}$$

where

$$I(\omega,x) = \int_x^\infty dx' \frac{\sin\omega(x-x')}{\omega}\overline{V}(x')e^{i\omega x'} \tag{5}$$

$$= \frac{-e^{-i\omega x}}{\alpha - 1}[(-2i\omega x_0)^{\alpha-2}\Gamma(2-\alpha) + \cdots]. \tag{6}$$

The omitted terms denote a convergent power series in ω. (This form is valid only for nonintegral α; the integral case can be obtained by taking a limit in the final result.) The factor $(-2i\omega x_0)^{\alpha-2}$ causes a cut in $\tilde{G}$ on the negative Imω axis. Thus, we have

$$G(x,y;t\to\infty) \simeq -\frac{f(0,x)f(0,y)}{g(0,0)^2}\frac{2^{\alpha-1}x_0^{\alpha-2}}{t^\alpha}. \tag{7}$$

The late-time behavior is therefore $\phi \simeq t^{-\alpha}$ generically [unless $\dot{\phi}(y,t=0) = 0$].

Notice that the time dependence is determined solely by the asymptotic form of V, while the magnitude, involving f and $g(0,0)$, is sensitive to the local geometry, and hence to the existence or otherwise of a horizon (in the full-line case). Note also that the Born approximation, which is strictly valid at very large x, has been used only to evaluate an x-independent Wronskian; therefore the results are exact at large t.

Next consider $l \neq 0$. It is necessary to handle the centrifugal barrier exactly, and to treat only $\overline{V}(x)$ by using the Born approximation. The zeroth order solutions are now Hankel functions rather than plane waves. A somewhat lengthy calculation, along the same lines as before, then leads to

$$G(x,y;t\to\infty) \simeq -\frac{f(0,x)f(0,y)}{g_0^2}\frac{C(l,\alpha)F(\alpha)}{t^{2l+\alpha}}, \tag{8}$$

where

$$C(l,\alpha) = \prod_{j=0}^{l-1}\frac{\alpha-2j-3}{\alpha+1+2j}, \qquad l = 1,2,\ldots, \tag{9}$$

and $C(l,\alpha) = 1$ for $l = 0$, with $F(\alpha) = 2(2x_0)^{\alpha-2}\Gamma(2l+\alpha)/\Gamma(\alpha)$, and $g_0 \equiv \lim_{\omega\to0}[(i\omega)^l W(g,f)]$, which is finite [12] and reduces to $g(0,0)$ for $l = 0$. The extra power of ω^l in the definition of g_0 is responsible for the suppression of the late-time tail by an extra factor of t^{-2l}, so that in general [unless $\dot{\phi}(y,t=0)$ vanishes], $\phi(x,t) \sim t^{-(2l+\alpha)}$ at late times. But there is an exceptional case: When α is an odd integer less than $2l+3$, $C(l,\alpha) = 0$ and the late-time tail vanishes in first order Born approximation;

higher order approximations representing multiple scatterings from asymptotically far regions give the next term going as $\sim t^{-(2l+2\alpha-2)}$.

Generically the late-time behavior is linear in the potential (first Born approximation). By applying $-\partial/\partial\alpha$ on the corresponding power-law potential, we obtain from Eq. (8) that for logarithmic potentials

$$G(x,y;t\to\infty) \simeq \frac{f(0,x)f(0,y)}{g_0^2}\frac{\partial}{\partial\alpha}\left[\frac{C(l,\alpha)F(\alpha)}{t^{2l+\alpha}}\right]. \quad (10)$$

The leading terms are $t^{-(2l+\alpha)}(c\ln t + d)$, except that $c \propto C(l,\alpha)$ vanishes when α is an odd integer less than $2l+3$. The Schwarzschild case ($\alpha = 3$) with $l \neq 0$ belongs to this exception; as is well known [2–6], the late-time behavior is a power law with exponent $-(2l+3)$, and no $\ln t$ factor.

Analytic results for ϕ at large t can then be obtained from Eqs. (7) and (8) with *no* adjustable parameters [except for case (c) in Fig. 1; see below], and are plotted as dashed lines in Figs. 1 and 2. The agreement is perfect. The exceptional cases (c) deserve mention. For case (c) in Fig. 1, the leading term vanishes, and the dashed line shown represents the next leading term arising from multiple scattering, whose time dependence is determined, but whose magnitude has been left as an adjustable normalization. For case (c) in Fig. 2, the vanishing of the leading term implies that the asymptotic slope should be zero (i.e., no $\ln t$, but only a pure power, whose magnitude is determined), and this indeed agrees with the numerical results, with no adjustable parameters. These results are for timelike infinity. Results for null infinity will be given in detail elsewhere.

In conclusion, we have achieved an analytic understanding of the late-time tail in such systems, with asymptotic formulas agreeing perfectly with numerical results. The late-time behavior is due to the tip of the cut in the frequency plane, which arises from scattering at large radius. For a potential that is a centrifugal barrier plus $\sim x^{-\alpha}(\ln x)^{\beta}$ ($\alpha > 2$, $\beta = 0, 1$), the late-time tail is generically $\sim t^{-(2l+\alpha)}(\ln t)^{\beta}$. The possibility of a logarithmic factor in the leading late-time behavior appears not to be widely known. Moreover, the case where α is an odd integer less than $2l+3$ is exceptional, and interestingly enough the most familiar Schwarzschild case belongs to this exception. We are at present extending our calculation to more general cases (e.g., when the potential has time dependence, using time-dependent perturbation theory).

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[9] For example, in a static spherically symmetric background, the evolution of linearized waves is given exactly by the KG equation (1) with the variable x related to the circumferential radius r by $-g_{tt}(dx)^2 = g_{rr}(dr)^2$, with t the Killing time.
[10] We assume here f' is nonzero at the origin. Otherwise, if $f \sim x^{l+1}$ as $x \to 0$, then we take $\lim_{x\to 0} x^{-(l+1)} f(\omega, x) = 1$ instead of $f' = 1$.
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