Solution to the Perturbative Infrared Catastrophe of Hot Gauge Theories

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Perturbative calculations of the free energy of a non-Abelian gauge theory at high temperature T break down at order g^6. This problem is solved by constructing a sequence of two effective field theories in which the momentum scales T and gT are integrated out. The free energy is decomposed into contributions from the momentum scales T, gT, and g^2T. The three terms are a power series in g^2, a power series in g beginning at order g^3, and a power series in g beginning at order g^6. In the g^2T contribution, the coefficients in the power series can be calculated using lattice simulations of 3-dimensional QCD. Renormalization group equations can be used to sum up the leading logarithms of $T/(gT)$ and of $gT/(g^2T)$.

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The perturbative infrared catastrophe of a non-Abelian gauge theory at high temperature is one of the most important unsolved problems in thermal field theory. The problem, which was first identified by Linde in 1979 [1,2], is that perturbative calculations of the free energy of a non-Abelian gauge theory at high temperature T with weak coupling g break down at order g^6. In contrast, there seems to be no obstacle to calculating the free energy of an Abelian gauge theory such as QED to arbitrarily high order in the coupling constant e. This problem casts a shadow over all applications of QCD perturbation theory to describe a quark-gluon plasma at high temperature, because the catastrophe afflicts any observable at sufficiently high order in g. The catastrophe also arises in applications of the electroweak gauge theory and grand-unified theories to describe phase transitions in the early Universe, since they also involve non-Abelian gauge theories at high temperature. The catastrophe can be avoided for static observables in hot QCD by calculating them directly using lattice simulations, but the computational resources that are required increase rapidly with T. Moreover, there are many problems, such as the calculation of dynamical observables or the calculation of static observables at nonzero baryon density, which are more easily addressed by perturbative methods than by lattice simulations. For these reasons, it is important to solve the problem of the perturbative infrared catastrophe.

In this Letter, we solve the problem by constructing a sequence of two effective field theories. The first effective theory, which is obtained by integrating out the momentum scale T, is equivalent to thermal QCD at length scales of order $1/(gT)$ or larger. The second effective theory, which is obtained by integrating out the scale gT from the first effective theory, is equivalent to thermal QCD at length scales of order $1/(g^2T)$ or larger. If T is sufficiently large, the parameters of the two effective theories can be calculated as perturbation series in the running coupling constant $g(T)$. The second effective theory is inherently nonperturbative, so that the effects from the scale g^2T must be calculated by lattice simulations.

To explain the origin of the infrared catastrophe, we first consider the case of the Abelian gauge theory QED. The leading term in the free energy is that of an ideal gas of electrons, positrons, and photons. The first correction due to the electromagnetic interaction is of order e^2, and can be calculated using ordinary perturbation theory. Beyond that order, infrared divergences arise in perturbation theory and it is necessary to sum infinite classes of diagrams. The order e^3 correction to the free energy can be calculated by resumming "ring diagrams," as in the classic calculation by Gell-Mann and Brueckner for a degenerate electron gas [3]. This method was used to calculate the order e^4 correction to the free energy of a degenerate electron gas in the relativistic limit [4]. An equivalent resummation method [5] was used to calculate the free energy to order g^4 for a massless scalar field with a ϕ^4 interaction in the high temperature limit [6]. More recently, this method was used to calculate the corrections to the free energy for hot QED of orders e^4 [7] and e^5 [8]. The only apparent obstacle to calculating to still higher order in e is the increasing complexity of the sums and integrals that are encountered.

We now consider the case of hot QCD. The resummation of ring diagrams allows the free energy to be calculated to order g^5. The free energy for pure-glue QCD was recently calculated to order g^4 by Arnold and Zhai [9] using an equivalent resummation method [10]. However, at order g^6, there are contributions to the free energy that can only be calculated using nonperturbative methods. To see where they arise, recall that static quantities in a thermal field theory can be calculated using the imaginary-time formalism, in which the propagator for a massless field is $1/(\omega_n^2 + \mathbf{k}^2)$, where ω_n takes on the discrete values $\omega_n = 2n\pi T$ for bosons and $\omega_n = (2n + 1)\pi T$ for fermions. The only modes that can propagate over distances much larger than $1/T$ are the $n = 0$ modes of the bosons. These are therefore the only modes that can give rise to

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infrared divergences that cause a breakdown of perturbation theory. With the resummation of ring diagrams, the chromoelectric zero modes ("electrostatic gluons") acquire a mass of order gT, while chromomagnetic zero modes ("magnetostatic gluons") remain massless. When restricted to the chromomagnetic zero modes, the action of QCD reduces to that of 3-dimensional Euclidean QCD with coupling constant g^2T. The perturbation expansion for this theory is hopelessly afflicted with infrared divergences, but it is well behaved nonperturbatively with a mass gap of order g^2T. Using simple dimensional analysis, one can show that the contribution to the free energy density from these modes is $(g^2T)^3T$ multiplied by a coefficient that can only be obtained by a nonperturbative calculation. There is no analogous contribution in QED, because the restriction to magnetostatic zero modes gives a free field theory.

We present the solution to the perturbative infrared catastrophe in the context of thermal QCD, although the solution applies equally well to any non-Abelian gauge theory. Our starting point is the imaginary-time formalism for thermal QCD, in which static quantities are calculated in 4-dimensional Euclidean QCD, with the periodic Euclidean time τ having period $\beta = 1/T$. The partition function is

$$Z_{\rm QCD}(T) = \int \mathcal{D}A_\mu^a \exp\left(-\int_0^\beta d\tau \int d^3x \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a\right), \tag{1}$$

where $G_{\mu\nu}^a$ is the non-Abelian field strength constructed out of the gauge field A_μ^a with coupling constant g. The free energy density is $F = -T \ln Z/V$, where V is the volume of space. It can be calculated at any temperature T using lattice simulations of the functional integral. If T is large enough that the running coupling constant $g(T)$ is small, one might expect to be able to calculate $F(T)$ as a power series in $g^2(T)$ using ordinary perturbation theory. However, this perturbation expansion breaks down due to infrared divergences associated with the exchange of static gluons. The divergences due to electrostatic gluons can be eliminated by the resummation of ring diagrams, but a more elegant solution is to construct an effective field theory which reproduces the static correlation functions of thermal QCD at distances of order $1/(gT)$ or larger. This theory, which we call electrostatic QCD (EQCD), contains an electrostatic gauge field $A_0^a(\mathbf{x})$ and a magnetostatic gauge field $A_i^a(\mathbf{x})$. Up to normalizations, they can be identified with the zero-frequency modes of the gluon field $A_\mu^a(\mathbf{x},\tau)$ for thermal QCD in a static gauge [11]. The Lagrangian for this effective field theory is

$$\mathcal{L}_{\rm EQCD} = \frac{1}{4} G_{ij}^a G_{ij}^a + \frac{1}{2}(D_i A_0)^a (D_i A_0)^a + \frac{1}{2} m_{\rm el}^2 A_0^a A_0^a + \delta\mathcal{L}_{\rm EQCD}, \tag{2}$$

where G_{ij} is the magnetostatic field strength with coupling constant g_E. The Lagrangian (2) has an SU(3) gauge symmetry. If the fields A_0 and A_i are assigned the scaling dimension 1/2, then the operators shown explicitly in (2) have dimensions 3, 3, and 1. The term $\delta\mathcal{L}_{\rm EQCD}$ in (2) contains all possible local gauge-invarient operators of dimension 2 and higher that can be constructed out of A_0 and A_i.

Renormalization theory tells us that static correlation functions at distances $R \gg 1/T$ in the full theory can be reproduced in the effective theory with any accuracy desired by tuning the coupling constant g_E, the mass parameter $m_{\rm el}^2$, and the parameters in $\delta\mathcal{L}_{\rm EQCD}$ as functions of T and the ultraviolet cutoff Λ_E of the effective theory [12]. The Λ_E dependence of the parameters is canceled by the Λ_E dependence of the loop integrals in the effective theory. The coefficients of the operators in the effective Lagrangian (2) for EQCD can be computed as perturbation series in $g^2(T)$. At leading order g, we have $g_E = g\sqrt{T}$ and $m_{\rm el}^2 = N_c g^2 T^2/3$. The coefficients of some of the higher-dimension operators were recently computed to leading order by Chapman [13]. Beyond leading order, it is particularly convenient to use a scale-invariant regularization scheme, such as dimensional regularization, which automatically eliminates all power ultraviolet divergences. In this case, the coefficient of an operator of dimension d is T^{3-d} multiplied by a power series in $g^2(T)$, with coefficients that are polynomials in $\ln(T/\Lambda_E)$. The parameters satisfy simple renormalization group equations that can be used to sum up leading logarithms of T/Λ_E.

The effective field theory EQCD can be used to calculate the partition function of thermal QCD, as well as static correlation functions at distance scales of order $1/(gT)$ or larger. The partition function is

$$Z_{\rm QCD}(T) = e^{-f_E(\Lambda_E)T^3V} \int^{(\Lambda_E)} \mathcal{D}A_0^a \mathcal{D}A_i^a \times \exp\left(-\int d^3x\, \mathcal{L}_{\rm EQCD}\right). \tag{3}$$

In the exponential prefactor, $f_E T^3$ can be interpreted as the coefficient of the unit operator which was omitted from the effective Lagrangian (2). It depends on the ultraviolet cutoff Λ_E of the effective theory in such a way as to cancel the cutoff dependence of the loop integrals in EQCD.

If we use dimensional regularization in $3-2\epsilon$ spatial dimensions to cut off both ultraviolet and infrared divergences, then f_E is given by the ordinary diagrammatic expansion for the free energy of thermal QCD. It can be calculated to next-to-next-to-leading order in g^2 by calculating 1-loop, 2-loop, and 3-loop diagrams without any resummation. These diagrams can be obtained from the results given in Ref. [9] by subtracting the effects of resummation. Taking the $\overline{\rm MS}$ (modified minimal subtraction) scale of dimensional regularization to be Λ_E and renormalizing the coupling constant at the scale $4\pi T$, we obtain

$$f_E = \frac{(N_c^2-1)\pi^2}{9}\left\{-\frac{1}{5} + \frac{N_c g^2(4\pi T)}{16\pi^2} + \left(\frac{N_c g^2}{16\pi^2}\right)^2\left[-\frac{12}{\epsilon} - 72\ln\frac{\Lambda_E}{4\pi T} - 4\gamma - \frac{116}{5} - \frac{220}{3}\frac{\zeta'(-1)}{\zeta(-1)} + \frac{38}{3}\frac{\zeta'(-3)}{\zeta(-3)}\right]\right\}, \quad (4)$$

where γ is Euler's constant and $\zeta(z)$ is the Riemann zeta function. The pole in ϵ arises from an infrared divergence. The mass parameter $m_{\rm el}^2$ in (2) is obtained to leading order in g^2 by matching the 1-loop propagator corrections for electrostatic gluons in thermal QCD to the propagator correction EQCD given by the term $m_{\rm el}^2 A_0^a A_0^a/2$ in the Lagrangian (2)

$$m_{\rm el}^2 = \frac{N_c g^2 T^2}{3}\left[1 + \epsilon\left(2\ln\frac{\Lambda_E}{4\pi T} + 2\frac{\zeta'(-1)}{\zeta(-1)}\right)\right]. \quad (5)$$

Since logarithmic ultraviolet divergences in the effective theory show up as poles in ϵ, we must keep the terms of order ϵ in (5).

Once the parameters in $\mathcal{L}_{\rm EQCD}$ have been determined to sufficiently high accuracy, the free energy density $F(T)$ can be calculated using lattice simulations of the functional integral in (3). If the coupling constant is sufficiently small, one might also expect to be able to calculate $F(T)$ using perturbation theory. The relevant expansion parameter is $g_E^2/m_{\rm el}$, which is of order g, so the perturbation series is an expansion in powers of $g(T)$, rather than $g^2(T)$. Unfortunately, the perturbation expansion of this effective field theory breaks down due to infrared divergences associated with magnetostatic gluons. It is convenient to construct a second effective field theory containing only the magnetostatic gauge field $A_i^a(\mathbf{x})$, which we call magnetostatic QCD (MQCD). The Lagrangian for this effective field theory is

$$\mathcal{L}_{\rm MQCD} = \frac{1}{4}G_{ij}^a G_{ij}^a + \delta\mathcal{L}_{\rm MQCD}, \quad (6)$$

where G_{ij} is the magnetostatic field strength with coupling constant g_M, which differs from g_E by perturbative corrections. The term $\delta\mathcal{L}_{\rm MQCD}$ includes all possible local gauge-invariant operators of dimension 5 and higher that can be constructed out of A_i^a. Renormalization theory tells us that magnetostatic correlation functions at distances $R \gg 1/(gT)$ or larger to EQCD can be reproduced in MQCD to any desired accuracy by tuning the coupling constant g_M and the parameters in $\delta\mathcal{L}_{\rm MQCD}$ as functions of the parameters of EQCD and the ultraviolet cutoff Λ_M of MQCD [12]. The partition function for thermal QCD can be expressed in terms of a functional integral in MQCD:

$$Z_{\rm QCD}(T) = e^{-f_E(\Lambda_E)T^3 V} e^{-f_M(\Lambda_E,\Lambda_M)(gT)^3 V} \times \int^{(\Lambda_M)} \mathcal{D}A_i^a \exp\left(-\int d^3x\,\mathcal{L}_{\rm MQCD}\right). \quad (7)$$

In the second prefactor, $f_M(gT)^3$ can be interpreted as the coefficient of the unit operator in the effective Lagrangian for MQCD, which was omitted in (6). Its dependence on the ultraviolet cutoff Λ_M is canceled by the Λ_M dependence of the loop integrals in MQCD.

If we use dimensional regularization to cut off both the ultraviolet and the infrared divergences in the perturbation expansion for EQCD, then $-f_M(gT)^3$ is just the logarithm of the partition function for EQCD. It can be obtained to next-to-leading order in g by calculating 1-loop and 2-loop diagrams in EQCD. Taking the $\overline{\rm MS}$ scale of dimensional regularization to be Λ_E, the result is

$$f_M(gT)^3 = \frac{N_c^2-1}{4\pi}m_{\rm el}^3\left\{-\frac{1}{3} + \frac{N_c g_E}{16\pi m_{\rm el}} \times \left[\frac{1}{\epsilon} + 4\ln\frac{\Lambda_E}{2m_{\rm el}} + 3\right]\right\}. \quad (8)$$

Note that the pole in ϵ in (8) cancels against the pole in $f_E T^3$, where f_E is given in (4). Furthermore, after making the substitution (5) for $m_{\rm el}$ in (8), the logarithms of Λ_E also cancel. The result is a logarithm of $T/m_{\rm el}$, whose coefficient was first obtained by Toimela [14].

In MQCD, perturbation expansions in the gauge coupling constant g_M are hopelessly plagued with infrared divergences. Thus the functional integral in (7) can only be calculated by nonperturbative methods, such as lattice simulations of MQCD. Surprisingly, f_G can still be expressed as a power series in $g(T)$. The most important momentum scale in MQCD is g_M, which is of order g^2T. The functional integral can therefore be expressed in the form

$$\int^{(\Lambda_M)} \mathcal{D}A_i^a \exp\left(-\int d^3x\,\mathcal{L}_{\rm MQCD}\right) \equiv e^{-f_G(\Lambda_M)(g^2T)^3 V}, \quad (9)$$

where f_G is dimensionless. This function f_G has a power series expansion in powers of $g(T)$. The leading term of order g^0 comes from dropping the correction terms $\delta\mathcal{L}_{\rm MQCD}$ in the effective Lagrangian. The resulting functional integral is

$$\int \mathcal{D}A_i^a \exp\left(-\int d^3x \frac{1}{4}G_{ij}^a G_{ij}^a\right) \equiv e^{-f_G^{(0)} g_M^6 V}. \quad (10)$$

Since a pure non-Abelian gauge theory in 3 dimensions has no logarithmic ultraviolet divergences, the only momentum scale is the gauge coupling constant g_M. Thus, by dimensional analysis, the coefficient $f_G^{(0)}$ is a pure number that can be calculated using lattice simulations of the functional integral (10). Higher order corrections to f_G can be calculated by treating the terms in $\delta\mathcal{L}_{\rm MQCD}$ as perturbations. The leading corrections come from the dimension-5 operators $g_M f^{abc} G_{ij}^a G_{jk}^b G_{ki}^c$ and $(D_i G_{jk})^a (D_i G_{jk})^a$, which have short-distance coefficients proportional to $g_E^2/m_{\rm el}^3$. The expectation values of these operators are of order g_M^{10}, but they are ultraviolet divergent and depend on the ultraviolet cutoff Λ_M of MQCD. This cutoff dependence is canceled by the Λ_M dependence of f_M and the parameters of $\mathcal{L}_{\rm MQCD}$. Thus the dimension-5 operators in $\delta \mathcal{L}_{\rm MQCD}$ give well-defined corrections to f_G that are of order g^3.

Combining (7) and (9), we obtain our final result for the free energy density of a non-Abelian gauge theory:

$$\frac{f_E(T)}{T} = f_E(\Lambda_E) T^3 + f_M(\Lambda_E, \Lambda_M)(gT)^3 + f_G(\Lambda_M)(g^2 T)^3, \quad (11)$$

where g is the running coupling constant $g(T)$. We have indicated the dependence of the functions on the arbitrary factorization scales Λ_E and Λ_M, which separate the momentum scales T, gT, and $g^2 T$. The coefficient f_E can be calculated using ordinary perturbation theory in thermal QCD in the form of a power series in $g^2(T)$, with coefficients that are polynomials in $\ln(T/\Lambda_E)$. The coefficient f_M can be calculated using perturbation theory in EQCD, which gives a power series $g(T)$ with coefficients that are polynomial in $\ln(\Lambda_E/gT)$ and $\ln(gT/\Lambda_M)$. The coefficient f_G can be expressed as a power series in $g(T)$, with coefficients that can be calculated using lattice simulations of MQCD. The functions f_E and f_M satisfy renormalization group equations that can be used to sum up leading logarithms of T/Λ_E and gT/Λ_M, respectively. By choosing Λ_E of order gT and Λ_M of order $g^2 T$, we can eliminate potentially large logarithms of $1/g$ from the perturbative expansions of f_M and f_G.

Adding (4) and (8), with $g_E = g\sqrt{T}$ and $m_{\rm el}$ given by (5), we recover the result for the free energy of pure-glue QCD to order g^4 that was recently obtained by Arnold and Zhai [9]. The order g^5 correction requires the calculation of $m_{\rm el}^2$ to next-to-leading order in g^2 and the calculation of the 3-loop diagrams for the partition function EQCD. A complete calculation to order g^6 requires calculating the 4-loop contribution to f_E, the 4-loop contribution to f_M, and the number $f_G^{(0)}$ defined by (10). This last number can only be obtained with a nonperturbative lattice calculation of the functional integral.

The problem of the perturbative infrared catastrophe of non-Abelian gauge theories is really one of providing a bridge between perturbative calculations of the effects of the large momentum scale T and the nonperturbative calculations that are required at the small momentum scale $g^2 T$. In this Letter, we have provided such a bridge by constructing a sequence of two effective field theories by first integrating out the momentum scale T and then integrating out the momentum scale gT. This method allows thermodynamic quantities like the free energy to be calculated with arbitrarily high accuracy in the high temperature limit.

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