# Stability of Chiral Luttinger Liquids and Abelian Quantum Hall States

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A criterion is given for topological stability of Abelian quantum Hall states, and of Luttinger liquids at the boundaries between such states; this suggests a selection rule on states in the quantum Hall hierarchy theory. The linear response of Luttinger liquids to electromagnetic fields is described: the Hall conductance is quantized, irrespective of whether edge modes propagate in different directions.

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A key feature of the quantum Hall effect (QHE) is its stability with respect to impurity potentials. A region of incompressible quantum Hall fluid described by a simple Laughlin state [1] has a current-carrying boundary where gapless excitations can be made. The edge is gapless because the current is carried by a single *chiral* "Luttinger liquid" mode [2]: stability against impurity perturbations may be associated with the *maximal chirality* of the edge. In contrast to the case of a nonchiral Luttinger liquid [3], there are no states of opposite chirality, so there is no backscattering (which can drive a "mass-generating" instability in which a pair of gapless modes with opposite chirality annihilate each other). In the integer QHE, there are a number of edge modes, but all have the same chirality, so the edge is stable. However, absolute stability (in the sense that there is *no* backscattering process) can occur even if the edge is *not* maximally chiral; in this Letter, I characterize the criterion for this more precisely. The property of edge stability also suggests a related stability property for "bulk" QHE states.

Recently, it has been suggested by Kane, Fisher, and Polchinski (KFP) [4] that when a "clean" Hall edge is not maximally chiral, the quantization of the Hall conductance generally breaks down (KFP also claim that, in certain—but not all—circumstances, "randomness at the edge" drives a novel non-mass-generating instability that preempts this). The coupling of non-maximally-chiral Luttinger liquids to electromagnetic fields is examined here, and, in contrast to [4], *no* breakdown of quantization of the Hall conductance of clean edges is found.

The essential ingredient of the effective low-energy theory of a quantum Hall state is a charged Abelian Chern-Simons (CS) gauge field [5–7] that couples to both the electromagnetic field and any other neutral degrees of freedom of the theory. In a wide class of possible QHE states (Abelian quantum Hall states), such as those of the hierarchy theory [8,9], the neutral degrees of freedom are also Abelian CS fields [10–13], and the Lagrangian density is given by

$$2\pi\mathcal{L} = \epsilon^{\lambda\mu\nu}\Big[\tfrac{1}{2}\,\hbar(\mathbf{a}_\lambda, \mathbf{K}\partial_\mu \mathbf{a}_\nu) + eA_\lambda(\mathbf{q}, \partial_\mu \mathbf{a}_\nu)\Big], \quad (1)$$

where $\mathbf{a}_\mu$ is an $n$-component vector of Abelian CS gauge fields, $A_\mu$ is the electromagnetic gauge field, $\mathbf{K}$ is a nonsingular *integer* coupling matrix (see, e.g., [13]), $\mathbf{q}$ is an *integer* vector, and $(\mathbf{a},\mathbf{b})$ is the inner product. This is a *topological field theory* (the Hamiltonian vanishes identically).

We may take $\mathbf{q}$ to be a multiple $q_0$ of a *primitive* integer vector (one where the components have no common factor). In a basis that separates off the neutral CS fields,

$$\mathbf{K} = \begin{pmatrix} k_0 & \mathbf{k}^T \\ \mathbf{k} & \mathbf{K}_0 \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} q_0 \\ \mathbf{0} \end{pmatrix}, \quad (2)$$

where $\mathbf{K}_0$ is a integer matrix of dimension $n-1$. Then $\sigma^H = \nu e^2/h$, with [10,13]

$$\nu = (\mathbf{q}, \mathbf{K}^{-1}\mathbf{q}) = \frac{q_0^2 \det K_0}{\det K}. \quad (3)$$

If the effective theory represents microscopic physics where electrons are the only mobile particles, $k_0 + q_0$ must be even, and $\mathbf{K}_0$ must be an *even integer matrix*, which means that all its diagonal elements are even. The structure of the theory is invariant under an equivalence transformation $\mathbf{K} \to \mathbf{W}\mathbf{K}\mathbf{W}^T$, $\mathbf{q} \to \mathbf{W}\mathbf{q}$, where $\mathbf{W}$ is an integer matrix with $|\det W| = 1$ [10].

It will be useful to introduce the notion of the *primitive form* of such a theory. If $\{\mathbf{K},\mathbf{q}\}$ is not in primitive form, $\det K$ has a factor $p^2 > 1$, and the theory is related to a primitive form $\{\tilde{\mathbf{K}}, \tilde{\mathbf{q}}\}$ by an integer matrix $\mathbf{W}$ with $\det W = p$, so that $\mathbf{K} = \mathbf{W}\tilde{\mathbf{K}}\mathbf{W}^T$, $\mathbf{q} = \mathbf{W}\tilde{\mathbf{q}}$, and $q_0 = \tilde{q}_0$. The total flux $\mathbf{\Phi}$ of the CS fields is $\int d^2r\, \vec{\nabla} \times \vec{\mathbf{a}}$; the effective theory allows vortex defects of the CS fields carrying flux $2\pi\mathbf{n}$, with $\mathbf{n}$ an integer vector, which have core energies determined by a Hamiltonian not specified in (1). The electric charge $Q$ is $e(\mathbf{q}, \mathbf{K}^{-1}\mathbf{n})$, and $|\det K|$ classes of vortices may be characterized by the value of $\mathbf{K}^{-1}\mathbf{n}$ modulo integer vectors. Vortices in the class where this vanishes have conventional statistics and integral $Q$ in units $q_0 e$. The theory is in primitive form if this is the *only* class with this property. The primitive form of a theory is equivalent to a nonprimitive form with a restriction on the allowed classes of vortices.

I now propose a definition of *topological stability* ($T$ stability): *An Abelian quantum Hall theory is $T$ stable if and only if $\mathbf{K}_0$ does not represent zero.* This means that there is no nontrivial solution of the equation $(\mathbf{m}, \mathbf{K}_0\mathbf{m}) = 0$, or, equivalently, of $(\mathbf{m}, \mathbf{K}\mathbf{m}) = (\mathbf{q}, \mathbf{m}) =$

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0. The physical idea motivating this is that if a theory is *not* $T$ stable, a pair of CS fields with opposite chirality decouple from the others and do not contribute to the QHE; backscattering between edge modes derived from these fields can then drive a mass-generating instability. A theory that is not $T$ stable has a primitive form

$$\mathbf{K} = \begin{pmatrix} 0 & 1 & \mathbf{0}^T \\ 1 & k & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{K}' \end{pmatrix}, \qquad \mathbf{q} = \begin{pmatrix} 0 \\ q_0 r \\ \mathbf{q}' \end{pmatrix}, \tag{4}$$

where $k = q_0 r (\text{mod} 2)$. This is because $\mathbf{K}$ can always be written as a symmetric tridiagonal matrix with positive off-diagonal elements, with $K_{11} = 0$ if it represents zero [14], and here $q_1 = 0$ also. If the theory is in primitive form, $K_{12} = 1$. A multiple of the first basis vector $|1\rangle$ (with which $\mathbf{K}$ represents zero) can be added to $|3\rangle$ so $K_{23}$ vanishes, and to $|2\rangle$ to reduce $K_{22}$ to 0 or 1. Then $\det K' = -\det K$, $\nu = (\mathbf{q}', \mathbf{K}'^{-1}\mathbf{q}')$, the signatures $\sigma(K)$ and $\sigma(K')$ are the same, and $(q_0)' = q_0 r'$ ($r$ and $r'$ integral).

At the edge of a region of incompressible Hall fluid, or more generally at the boundary between Hall fluids with different $\sigma^H$, there must be a charge-non-conserving edge current with an anomaly in its continuity equation,

$$\partial_t \rho(x) + \partial_x j(x) = \sigma^H E_x, \tag{5}$$

where $x$ is the coordinate along the edge, and $\sigma^H = (e^2/h)\Delta\nu$ is now the *change* in the Hall constant across the edge. This compensates the jump in the Hall current normal to the edge ($E_x$ is continuous, but the Hall constant is not). The edge between Abelian Hall fluids is a Luttinger liquid [2], also characterized by a $\{\mathbf{K}, \mathbf{q}\}$ pair, which is that of the bulk CS theory if the edge is between a QHE state and a nonconducting region.

The Luttinger liquid is described in terms of $n$ fields $\varphi_i(x)$, with equal-time commutation relations

$$[\varphi_i(x), \varphi_j(x')] = i\pi[K_{ij}\,\text{sgn}(x - x') + L_{ij}], \tag{6}$$

where $L_{ij} = \text{sgn}(i - j)(K_{ij} + q_i q_j)$ is a Klein factor. Let $\varphi_{\mathbf{m}}$ be $\sum_i m_i \varphi_i$, where $\mathbf{m}$ is an integer vector. A set of integrally charged, fermionic, or bosonic *local fields* are given by

$$\Psi_{\mathbf{m}}(x) = e^{-i\varphi_{\mathbf{m}}}, \qquad [Q, \Psi_{\mathbf{m}}] = eq(\mathbf{m})\Psi_{\mathbf{m}}, \tag{7}$$

with $q(\mathbf{m}) = (\mathbf{q}, \mathbf{m})$. These local fields are also characterized by the integer quadratic form $K(\mathbf{m}) = (\mathbf{m}, \mathbf{K}\mathbf{m})$ where $(-1)^{K(\mathbf{m})} = (-1)^{q(\mathbf{m})}$, and are fermionic or bosonic depending on whether $K(\mathbf{m})$ is odd or even: $\Psi_{\mathbf{m}}(x)\Psi_{\mathbf{m}'}(x') = (-1)^{q(\mathbf{m})q(\mathbf{m}')}\Psi_{\mathbf{m}'}(x')\Psi_{\mathbf{m}}(x)$ for $x \neq x'$ (the Klein factor ensures this when $\mathbf{m} \neq \mathbf{m}'$).

The gauge-invariant electric charge density is

$$\rho = \frac{e}{2\pi}\sum_{ij} q_i K_{ij}^{-1} D_x \varphi_j, \tag{8}$$

where $D_x\varphi_i$ is the covariant derivative $(\partial_x \varphi_i - q_i e A_x/\hbar)$, and $A_x(x)$ is the component of the electromagnetic vector potential parallel to the edge. The charge density operator has the anomalous commutation relation $[\rho(x), \rho(x')] = i\hbar\sigma^H\delta'(x - x')$, which gives rise to the anomaly in the continuity equation. The effective Hamiltonian is $H = H_{\text{LL}} + \int dx\, \phi\rho$, where $\phi(x)$ is the electrostatic potential, and $H_{\text{LL}} = H_0 + H_1$ is the harmonic effective Luttinger liquid Hamiltonian,

$$H_0 = \frac{1}{2\pi}\int dx\, \epsilon_i(x) D_x \varphi_i(x), \tag{9}$$

$$H_1 = \frac{1}{4\pi}\int dx \int dx'\, \mathcal{V}_{ij}(x, x') D_x \varphi_i(x) D_{x'} \varphi_j(x'). \tag{10}$$

It will be convenient to consider the case where $H_{\text{LL}}$ is translationally invariant, and long-range forces are screened, so that $\epsilon_i(x) = \hbar\omega_i$, constant, and $\mathcal{V}_{ij}(x, x') = \hbar V_{ij}\delta(x - x')$, where $V_{ij}$ is a positive-definite matrix with the dimensions of velocity.

The gauge-invariant edge current is

$$j = -\frac{\delta H_{\text{LL}}}{\delta A_x} = \frac{e}{2\pi}\sum_{ij} q_i(\omega_i \delta_{ij} + V_{ij} D_x \varphi_j), \tag{11}$$

and is not conserved if $\sigma^H \neq 0$. In that case, it is also possible to define a conserved but non-gauge-invariant edge charge density and current by $\rho^{\text{tot}} = \rho + \sigma^H A_x$, $j^{\text{tot}} = j + \sigma^H \phi$; only the *full* theory combining the chiral Luttinger liquid with the CS fields of the bulk Hall fluids is both gauge invariant *and* current conserving.

When diagonalized, $H_{\text{LL}}$ describes edge modes propagating with velocities $v_\lambda$ obtained by solution of the generalized real symmetric eigenvalue problem

$$\mathbf{V}\mathbf{u}_\lambda = v_\lambda \mathbf{K}^{-1}\mathbf{u}_\lambda, \tag{12}$$

which has real eigenvalues and linearly independent eigenvectors when $\mathbf{V}$ is positive definite. It is convenient to define the orthogonal matrix $\mathbf{O}$ by

$$\mathbf{V}^{1/2}\mathbf{K}\mathbf{V}^{1/2} = \mathbf{O}\mathbf{v}_d\mathbf{O}^T, \tag{13}$$

where $\mathbf{v}_d$ is the diagonal matrix $\text{diag}(v_1, \ldots, v_n)$; note that its signature is given by $\sigma(v_d) = \sigma(K)$.

Since the effective Hamiltonian is harmonic, it is straightforward to compute the linear response of the gauge-invariant edge charge density and current to changes $\delta\phi(x,t)$, $\delta A_x(x,t)$ in the electromagnetic potentials (Kubo formula). In the thermodynamic limit, $\delta\langle j(x,t)\rangle$ is given by

$$\int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} dx'\, \sigma^{xx}(x - x', t - t')\delta E_x(x', t'), \tag{14}$$

and $\delta\langle\rho\rangle$ is given by a similar expression with a kernel $\sigma^H(x,t)$, where (independent of temperature)

$$\begin{aligned} \sigma^{xx}(x,t) &= \frac{e^2}{h}\sum_\lambda (a_\lambda)^2 \delta(x - v_\lambda t), \\ \sigma^H(x,t) &= \frac{e^2}{h}\sum_\lambda (a_\lambda)^2 v_\lambda^{-1} \delta(x - v_\lambda t), \end{aligned} \tag{15}$$

with $a_\lambda = \sum_i q_i (V^{1/2}O)_{i\lambda}$. Then $\sigma^H(x,0) = \sigma^H \delta(x)$ and $\sigma^{xx}(x,0) = \gamma_D \delta(x)$ where $\gamma_D = (e^2/h)(\mathbf{q}, \mathbf{Vq})$ is the Drude weight, which characterizes the commutation relation $[\rho(x), j(x')] = i\hbar \gamma_D \delta'(x - x')$.

Now consider

$$\int_0^\infty dt\, \sigma^{xx}(x,t) = G + \tfrac{1}{2}\sigma^H \operatorname{sgn}(x), \qquad (16)$$

where $G = \frac{1}{2}(e^2/h)(\mathbf{q}, \mathbf{\Lambda}^{-1}\mathbf{q})$ and

$$\mathbf{V}^{1/2}\mathbf{\Lambda}\mathbf{V}^{1/2} = \mathbf{O}|\mathbf{v}_d|\mathbf{O}^T, \qquad (17)$$

with $|\mathbf{v}_d| \equiv \operatorname{diag}(|v_1|, \ldots, |v_n|)$. When $\sigma^H = 0$, $G$ is the *conductance* [15]: In steady state, a uniform current $\delta\langle j\rangle = G \int dx\, \delta E_x(x)$ flows in response to an applied potential drop along the system. This is *not* so when $\sigma_H \neq 0$, *because of the anomaly* (5) *which does not permit a uniform steady-state current unless* $\delta E_x$ *vanishes*: Hall currents flow onto the boundary and build up a charge distribution that cancels the externally applied electric field, so that the *total* field $\delta E_x$ [which is what enters in the Kubo formula (14)] vanishes in steady state. Thus the Hall edge adjusts itself to lie along an equipotential, and the steady-state gauge-invariant current $\delta\langle j\rangle$ vanishes. The conserved but non-gauge-invariant current is then constant, and given by $\delta\langle j^{\text{tot}}\rangle = \sigma^H \delta\phi$, as expected.

As a formal quantity, $G$ satisfies the inequality $G \geq \frac{1}{2}|\sigma_H|$, which is an equality when $a_\lambda$ is only nonzero for modes $\lambda$ of one chirality. In general, $G$ depends on $\mathbf{V}$ and is nonuniversal; however, the following analysis shows that $G$ does *not* affect the Hall conductance.

The expression (11) is not valid at points $x_i$ where currents $\delta i_i$ flow onto the Hall edge through tunneling contacts [in the absence of a microscopic description of the contacts, a Kubo formula for $\delta\langle j(x,t)\rangle$ can only be used when $x \neq x_i$]. In steady state, the contact points $x_i$ are the only places that give nonzero contributions $-\delta V_i$ to $\int dx\, \delta E_x$. The Kubo formula gives $\delta i_i = \delta j(x_i^+) - \delta j(x_i^-) = \sigma^H \delta V_i$ *independent of* $G$. Note that $\delta V_i$ is the potential jump across the contact *along the Hall edge*, *not* the potential difference between the edge and the reservoir from which the tunneling current is drawn, and *does not allow the terminal conductance of the contact to be deduced.* If $x_1 < x_2 < x_3 < x_4$, and a measured current $\delta i$ flows onto the edge at $x_1$, and off it at $x_3$, a potentiometer (drawing no current) connected between $x_2$ and $x_4$ will measure a *quantized* Hall conductance $\delta i/\delta V_{24} = \sigma^H$. This analysis assumes that the separation between contacts (and perhaps their size) is large compared to the short-distance cutoff scale above which the effective Luttinger liquid theory is valid.

The *scaling dimension* of the field $\Psi_\mathbf{m}$ is defined by the multiplicative renormalization when it is normal ordered with respect to the ground state of $H_{\text{LL}}$. If periodic boundary conditions on a length $L$ are imposed, $:\Psi_\mathbf{m}: = (L/\xi)^{\Delta(\mathbf{m})}\Psi_\mathbf{m}$, where $\Delta(\mathbf{m}) = \frac{1}{2}(\mathbf{m}, \mathbf{\Lambda m})$ ($\xi$ is a short-distance cutoff). From (13) and (17), $\Delta(\mathbf{m}) \geq \frac{1}{2}|K(\mathbf{m})|$.

Now consider stability of the Luttinger liquid with respect to perturbations of the form

$$H' = \int dx\, [t(x)\Psi_\mathbf{m}(x) + t^*(x)\Psi_{-\mathbf{m}}(x)]. \qquad (18)$$

Three cases can be considered: (a) *clean* case, $t(x) = |t(x)|\exp[i\alpha(x)]$ where $|t(x)|$ and $\partial_x \alpha(x)$ are constant; (b) *random* case, they vary randomly, with $\langle |t(x)|^2\rangle$ finite; and (c) *local* case, $|t(x)|$ vanishes except near a point $x_0$.

Clearly $\Psi_\mathbf{m}$ must be bosonic and charge conserving, with $q(\mathbf{m}) = 0$ and even $K(\mathbf{m})$. In the interaction picture, the vacuum amplitude

$$U(L,T) = \langle 0|T_t \exp\left[-i\hbar^{-1}\int_0^T dt\, H'(t)\right]|0\rangle \qquad (19)$$

can be expanded in powers of $t(x)$,

$$\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \prod_{i=1}^{n} \sum_{s_i=\pm 1} \int_0^L dx_i \int_0^T dt_i\, A(\{x_i, t_i, s_i\}), \qquad (20)$$

where $(x_i, t_i)$ are the space-time coordinates of a $s_i = \pm 1$ tunneling event (or "instanton") at which $\Psi_{\pm\mathbf{m}}$ acts, with $\sum_i s_i = 0$. The amplitude $A$ is a product of a *one-event* factor $\prod_i A_1(x_i, s_i)$ and a *two-event* factor $\prod_{i<j} A_2(x_{ij}, t_{ij}, s_{ij})$, with $x_{ij} \equiv x_i - x_j$, $t_{ij} \equiv t_i - t_j$, and $s_{ij} \equiv s_i s_j$. Here $A_1 = [|t(x_i)|/\hbar]\exp[i s_i \overline{\alpha}(x_i)]$, where $\overline{\alpha}(x) = \alpha(x) - Q_0(\mathbf{m})x$, with $Q_0(\mathbf{m}) = \langle 0|\partial_x \varphi_\mathbf{m}|0\rangle$ and

$$A_2 = \prod_\lambda \left[\frac{i d(x_{ij} - v_\lambda t_{ij})}{\xi \operatorname{sgn}(v_\lambda)}\right]^{\eta_\lambda(\mathbf{m}) s_{ij}}, \qquad (21)$$

where $\eta_\lambda(\mathbf{m}) = |v_\lambda|[\mathbf{O}^T\mathbf{V}^{-1/2}(\mathbf{m}\mathbf{m}^T)\mathbf{V}^{-1/2}\mathbf{O}]_{\lambda\lambda} \geq 0$, with $\sum_\lambda \eta_\lambda(\mathbf{m}) = 2\Delta(\mathbf{m})$ and $\sum_\lambda \eta_\lambda(\mathbf{m})\operatorname{sgn}(v_\lambda) = K(\mathbf{m})$; $|d(x_{ij} - v_\lambda t_{ij})| \gg \xi$, where $d(x) \equiv (L/\pi)\sin(\pi x/L)$ , is assumed in (21).

Perturbations with $K = 0$ are mass generating when relevant (the sine-Gordon model is a familiar example). In the clean case, the complex phase of the factor $A_1$ prevents relevance of $H'$ unless $Q_0 = \partial_x \alpha$. For relevance, $A_2$ requires the scaling dimension $\Delta$ to be sufficiently small. The critical values of $\Delta$ follow from the behavior of (20) if $L$ and $T$ are rescaled: $\Delta < 2$ (clean case), $\Delta < \frac{3}{2}$ (random case), and $\Delta < 1$ (local case) [16,17].

When the $K = 0$ perturbation is relevant, the pair of modes that "split off" in (4) become a massive degree of freedom that is removed from the low-energy theory. The physical significance of the condition $K = 0$ is that it implies $[\partial_x \varphi_\mathbf{m}(x), \Psi_{\pm\mathbf{m}}(x')] = 0$. When the perturbation is relevant, the phase gradient $\partial_x \varphi_\mathbf{m}$ becomes rigidly locked to the value $\partial_x \alpha$ at large length scales. In the random case, rigidity occurs in finite domains separated by pinned "domain walls" (solitons), and in the local case, occurs only near $x_0$. This rigidity can only develop if the perturbation commutes with the phase gradient.

The only other perturbations that could be relevant ($\Delta < 2$) have $|K| = 2$, but are *not* mass generating; KFP

[4] argue that, in the random case, a $|K| = 2$ perturbation with $\Delta < \frac{3}{2}$ renormalizes $\mathbf{V}$ so $\Delta \rightarrow 1$, its minimum value. KFP's arguments seem based in part on the familiar renormalization group (RG) treatment of the $K = 0$ case, but the *clean* $|K| = 2$ case [18] seems to behave very differently, raising doubts about such analogies.

If a (primitive) $\mathbf{m}$ with $K(\mathbf{m}) = q(\mathbf{m}) = 0$ exists, the stability of the Luttinger liquid depends on the nonuniversal Hamiltonian coupling matrix $\mathbf{V}$, which determines $\Delta(\mathbf{m})$. On the other hand, if the matrix $\mathbf{K}_0$ *does not represent zero*, the edge is absolutely stable ($T$ stable), independent of $\mathbf{V}$. If $\mathbf{K}$ is a definite matrix $[|\sigma(K)| = n]$, the edge is maximally chiral and $T$ stable. This is sufficient but *not* necessary: the condition given here characterizes $T$ stability in its most general form.

Clearly, if $\mathbf{K}_0$ is definite, $T$ stability is present, as in the case of the $\nu = 2/3$ edge considered in [4]. If $\mathbf{K}_0$ is *indefinite* (but $\sigma^H \neq 0$, so it is nonsingular), its dimension $n_0 = n - 1 \geq 2$. If $n_0 = 2$, the system is $T$ stable if and only if $-\det K_0$ is not a perfect square [14]. On the other hand, if $n_0 > 4$, an indefinite $\mathbf{K}_0$ *always* represents zero [14]. If $n_0$ is 3 or 4, an indefinite $\mathbf{K}_0$ does not represent zero if it fails to represent zero "$p$-adically" [14] when $p$ is 2, or any odd prime factor of $\det K_0$; this property depends on the *rational equivalence class* of $\mathbf{K}_0$, and can be quickly tested by a finite calculation *without* searching for a vector $\mathbf{m}$ with which $\mathbf{K}_0$ represents zero.

The formalism presented here allows the analysis of the edge between Abelian Hall states $\{\mathbf{K}^A, \mathbf{q}^A\}$ and $\{\mathbf{K}^B, \mathbf{q}^B\}$. The first step is to form the direct sum $\mathbf{K} = \mathbf{K}^A \oplus (-\mathbf{K}^B)$, $\mathbf{q} = \mathbf{q}^A \oplus \mathbf{q}^B$, and reduce the edge theory to a primitive form. At an edge between Abelian Hall states with the *same* Hall constant, the Luttinger liquid has $\sigma^H = 0$, and $\mathbf{K}$ can be put into the form (4), with $\mathbf{q}' = \mathbf{0}$. A gap for making charged excitations can open, leaving a residual neutral gapless Luttinger liquid. A RG treatment of the instabilities at edges between various Abelian Hall states will be presented elsewhere [18].

The parameters $\mathbf{V}$ of the Luttinger liquid that determine $\Delta(\mathbf{m})$ are determined by the internal structure of the edge, and can change through nonlinear deformations. It is tempting to speculate that if an edge can exhibit an instability that reduces the number of edge modes, it will relax so that the instability occurs, so only a $T$-stable edge is robust. Since $T$-stable theories are so stable, it seems natural to conjecture that only Abelian Hall states with this property occur in nature.

In the hierarchy theory, $q_0 = 1$, and $\mathbf{K}_0$ can be represented [8,10] as a tridiagonal matrix with off-diagonal elements all taking the value 1, and diagonal elements $(p_1, p_2, \ldots, p_{n-1})$, where $p_i$ are nonzero even integers. At each level of the hierarchy, the most stable condensates of quasiparticles correspond to $|p_i| = 2$, and are stabilized by contact repulsion (these may be called the *principal* daughter states).

The $T$-stability condition can now be applied to determine the allowed *signs* that the $p_i$ can have. First, $\pm(2,2,2,\ldots)$ represents the diagonal elements when $\mathbf{K}_0$ is definite. When it is indefinite, $\pm(2,-2)$ is stable; for $n - 1 > 2$ the test [14] shows that the *only* $T$-stable case is $\pm(2,-2,2)$. In the hierarchy theory, these correspond to states with $\nu^{-1} = k_0 \mp \frac{2}{5}$ and $k_0 \mp \frac{5}{12}$, with $k_0$ odd. Thus in the range $\frac{1}{3} \leq \nu < \frac{1}{2}$, the *only* topologically stable families of spin-polarized (or one-component) hierarchy states with principal descendants are the familiar infinite-length sequence $(\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \ldots)$ and the length-4 sequence $(\frac{1}{3}, \frac{2}{5}, \frac{5}{13}, \frac{12}{31})$. Observation of the finite sequence would test the idea that the physically realized states are restricted to topologically stable (and usually) principal sequences of hierarchy states.

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