Spontaneous Symmetry Breaking in a One Dimensional Driven Diffusive System

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A simple model of a driven diffusive system which exhibits spontaneous symmetry breaking in one dimension is introduced. The model has short range interactions and unbounded noise. It is characterized by an asymmetric exclusion process of two types of charges moving in opposite directions on an open chain. The model is studied by mean field and Monte Carlo methods. Exact solutions can be found in a restricted region of its parameter space. A simple physical picture for the symmetry breaking mechanism is presented.

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The question of spontaneous symmetry breaking and long range order in one dimensional (1D) systems with short range interactions and small but unbounded noise is an intriguing one. It is well known that under these conditions no phase transition takes place in thermal equilibrium. This is the case provided the local variable which describes the state of the system can take only a finite number of possible values, such as in the Ising or the Potts models. When the local variable is not restricted to a finite set of values, such as in solid-on-solid models for chain unbinding, phase transitions, and symmetry breaking may take place [1]. Systems far from thermal equilibrium are, on the other hand, less restrictive, and the question of whether they are capable of exhibiting spontaneous symmetry breaking under the above conditions even when the local variable can take only a finite set of values has been open for quite some time. Recently, an example of such a phase transition, in the context of error correcting computation algorithms, has been given [2]. However, this example is rather complicated and not widely understood.

In the present Letter we introduce a simple nonequilibrium one dimensional model with short range interactions and unbounded noise which exhibits spontaneous symmetry breaking in the thermodynamic limit. The local dynamical variable associated with the model is restricted to take only a finite number of possible states. The model belongs to a class of traffic jam models or driven diffusive systems. It may also describe the dynamics of a certain growth process [3]. To be specific we consider a 1D lattice of length N. Each lattice point may be occupied by either a positive (+) or a negative (−) particle, or by a hole (0). The (+) particles move to the right while the (−) particles move to the left. The two kinds of particles may pass each other. The positive (negative) particles are supplied at the left (right) end and removed at the right (left) end of the system. The model possesses a right-left symmetry, and the dynamical rules are invariant under charge conjugation combined with space inversion. Under the conditions where the symmetry of the dynamical process is preserved, one expects the two currents of the positive and the negative charges to be equal. When spontaneous symmetry breaking takes place, the two currents become unequal in the thermodynamic limit (defined below).

We now define more precisely the dynamics of the system. During an infinitesimal time interval dt, the following exchange events take place between two adjacent sites:

$$+0 \rightarrow 0+, \quad 0- \rightarrow -0, \quad +- \rightarrow -+, \tag{1a}$$

with probabilities dt, dt, and $q\,dt$, respectively. Furthermore, at the two ends, particles may be introduced or removed. At the left boundary ($i = 1$) one has

$$0 \rightarrow +, \quad - \rightarrow 0, \tag{1b}$$

with probabilities $\alpha\,dt$ and $\beta\,dt$, respectively. Similarly at the right boundary ($i = N$):

$$0 \rightarrow -, \quad + \rightarrow 0, \tag{1c}$$

with probabilities $\alpha\,dt$ and $\beta\,dt$, respectively. The dynamical process in the bulk is conservative: It conserves both the positive and the negative charges. However, these quantities are not conserved at the two ends.

One is interested in the steady state calculating, say, the density profiles of the two charges and the corresponding currents. The model defined above is a generalization of the totally asymmetric exclusion model of a single type of particles [4–7]. The bulk dynamics (1a) of two species but with periodic boundary conditions has been studied in connection with the behavior of shock fronts [8]. Related models in higher dimensions have also been considered recently [9,10].

In the present work we make the observation that the model (1) exhibits spontaneous symmetry breaking for a certain range of the parameters α, β, q which define its dynamics. We find two phases in which the currents

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corresponding to the positive and negative charges are not equal in the thermodynamic limit. In each of these phases the system may be in either one of two states related to each other by charge conjugation and space inversion $i \rightarrow N - i + 1$. Symmetry breaking does not take place in the single species model.

The model (1) is exactly soluble for $\beta = 1$ and in the limit $\alpha \rightarrow \infty$. The steady state profiles and the currents may be obtained in this case by the recently introduced matrix method [5,8]. However, the phases are symmetric for this set of parameters (for details see [3]).

In order to obtain the qualitative features of the global phase diagram we first study the model within the mean field approximation, where two broken symmetry phases are found for sufficiently small β. We then carry out numerical simulations of the dynamical equations in which the predictions of the mean field approximation, and, in particular, the existence of broken symmetry phases, are substantiated. A simple argument supporting these findings is given. This Letter is concluded by a study of the time scale associated with flipping between the two states of a broken symmetry phase for a finite system.

To study the mean field phase diagram of the model we denote the density of the positive and the negative charges at site i by p_i and m_i, respectively. Within the mean field approximation one neglects density-density correlations and obtains the following equations for the steady state:

$$\begin{aligned} J_+ &= p_i[1 - p_{i+1} - (1 - q)m_{i+1}], \\ J_- &= m_{i+1}[1 - m_i - (1 - q)p_i], \end{aligned} \tag{2a}$$

for $i = 1, \ldots, N - 1$, where J_+ and J_- are the currents of the positive and negative charges, respectively. We have used the fact that, in the steady state, the currents $J_\pm$ are independent of position. In addition to the bulk equations (2a) one has four other equations for the currents at the boundaries,

$$\begin{aligned} J_+ &= \alpha(1 - p_1 - m_1) = \beta p_N, \\ J_- &= \beta m_1 = \alpha(1 - p_N - m_N). \end{aligned} \tag{2b}$$

For simplicity we discuss the resulting phase diagram for the case $q = 1$ (the qualitative results remain unchanged for $q \neq 1$). It is readily seen from Eqs. (2) that when $q = 1$ the two sets of bulk equations decouple. The reason is that, away from the boundaries, a positive particle does not distinguish between a hole and a negative particle and neither does a negative particle distinguish between a hole and a positive particle. However, at the boundaries the two systems of particles are coupled via the boundary equations (2b). Defining

$$\begin{aligned} \alpha^+ &= \alpha h_1/(1 - p_1) = J_+/(J_+/\alpha + J_-/\beta), \\ \alpha^- &= \alpha h_N/(1 - m_N) = J_-/(J_-/\alpha + J_+/\beta), \end{aligned} \tag{3}$$

where $h_i = 1 - p_i - m_i$ is the hole density at site i, the problem is reduced to two one-species totally asymmetric exclusion processes on a lattice of size N. One process corresponds to the (+) particles with boundary parameters (α^+, β) and the other corresponds to the (−) particles with boundary parameters (α^-, β). The only coupling between the two processes is via the boundary equations (3).

The phase diagram for the one-species process with boundary parameters (α^s, β^s) is known [4–6] (here s stands for single species). It exhibits three phases: (a) a power law phase for $\alpha^s \geq \frac{1}{2}, \beta^s \geq \frac{1}{2}$. In this phase the approach to the bulk density $(= \frac{1}{2})$ is algebraic, and the current is maximal $(J^s = \frac{1}{4})$. (b) A low density phase for $\alpha^s < \beta^s$ and $\alpha^s < \frac{1}{2}$. Here the approach to the bulk density $(= \alpha^s)$ is exponential, and the current is $J^s = \alpha^s(1 - \alpha^s)$. (c) A high density phase for $\beta^s < \alpha^s$ and $\beta^s < \frac{1}{2}$, in which the approach to the bulk density $(= 1 - \beta^s)$ is exponential and the current is $J^s = \beta^s(1 - \beta^s)$. The high and low density phases coexist on the line $\alpha^s = \beta^s < \frac{1}{2}$. Using these results and Eq. (3) to get the effective feeding parameters α^+, α^-, the (α, β) phase diagram of the model (1) and closed expressions for the transition lines may be obtained (for details see [3]). The phase diagram (Fig. 1) exhibits four phases of which two are symmetric and two are nonsymmetric. One of the symmetric phases is characterized by a power law decay of the local density, and the other is a low density phase. The two nonsymmetric phases are characterized by high density–low density (hd-ld) and low density–low density (ld-ld) profiles, and they exist in the low β region of the phase diagram. In these phases the currents of the positive and the negative charges are unequal.

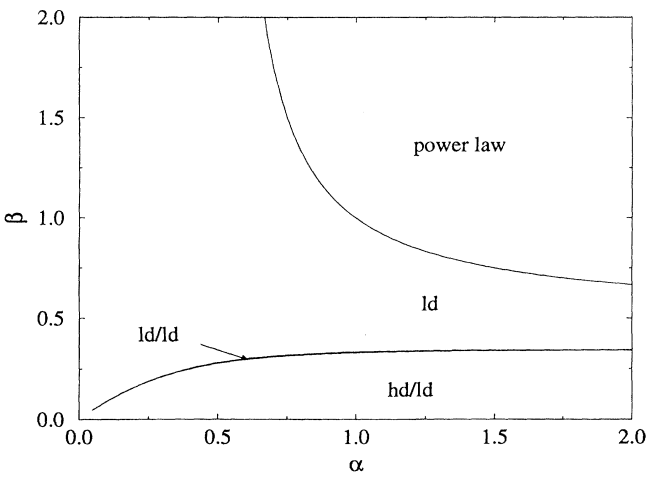

FIG. 1. The (α, β) mean field phase diagram for $q = 1$. It exhibits two symmetric phases: power law and low density (ld), and two nonsymmetric phases: low density–low density (ld-ld) and high density–low density (hd-ld). The ld-ld phase occupies a narrow region which appears as a line on the scale of the figure. The transitions between the various phases are continuous.

To demonstrate that spontaneous symmetry breaking does exist in the stochastic model (1), we carried out extensive Monte Carlo (MC) simulations. Figure 2 shows typical density profiles in the two nonsymmetric phases. They are obtained by averaging the occupations of each site over the simulation. To calculate these profiles one has to run the MC simulations long enough to reach the steady state. However, running time should be smaller than $\tau(N)$, the characteristic flipping time between the two states of the broken symmetry phase. The density profiles given in Fig. 2 are flat in the bulk, with some structure near the ends. In the hd-ld phase, the density of the negative charges is larger than $\frac{1}{2}$, while the density of the positive charges is lower than $\frac{1}{2}$. In the ld-ld phase both densities are smaller than $\frac{1}{2}$.

Consider now the characteristic time $\tau(N)$ between flips. It has to diverge in the thermodynamic limit in order to have a stable broken symmetry phase. Moreover, since the bulk dynamics of this model is conservative, even if $\tau(N)$ grows like N^2 for large N it may not be sufficient to demonstrate spontaneous symmetry breaking. For example, if one considers the 1D Ising model with conserved bulk dynamics but with some nonconserved dynamics at the boundaries, the characteristic time associated with the decay of magnetization grows like N^2. This divergence is a result of the slow conserved dynamics, and does not indicate spontaneous symmetry breaking. The thermodynamic limit is thus taken to be the large t and N limit where $O(N^2) < t < O(\tau(N))$.

In the present model $\tau(N)$ is expected to grow much faster than N^2. This can be seen as follows. Let the broken symmetry phase be characterized by currents j_1, j_2, and bulk densities ρ_1, ρ_2 of the two species. In the two symmetry related states, denoted by A and B, the currents and the bulk densities are given by $J_+ = j_i, \rho_+ = \rho_1$; $J_- = j_2, \rho_- = \rho_2$ in state A, and $J_+ = j_2, \rho_+ = \rho_2$; $J_- = j_1, \rho_- = \rho_1$ in state B. Here $\rho_\pm$ are the bulk densities of the positive and negative charges, respectively. The question is how does a finite system flip from, say, state A to state B? Clearly, a change in the density throughout the lattice has to be induced by fluctuations at the boundaries, since the bulk dynamics is conservative. Suppose such a fluctuation takes place and a droplet of state B is generated near one of the ends of a system of state A. This droplet is separated from the rest of the system by a domain wall. However, unlike a domain wall separating two equivalent states in thermal equilibrium (such as a domain wall between the up and the down states in the Ising model), here the domain wall is not stable [11]. The reason is that the two currents of, say, the positive charges in the two coexisting states, j_1 and j_2, are different from each other and there is a net flux of particles into or out of the domain wall region. As a result, the droplet B is expelled and the system relaxes back to the A state on a time scale which depends on the length of the initial B droplet. The system eventually flips to state B by a mechanism explained below.

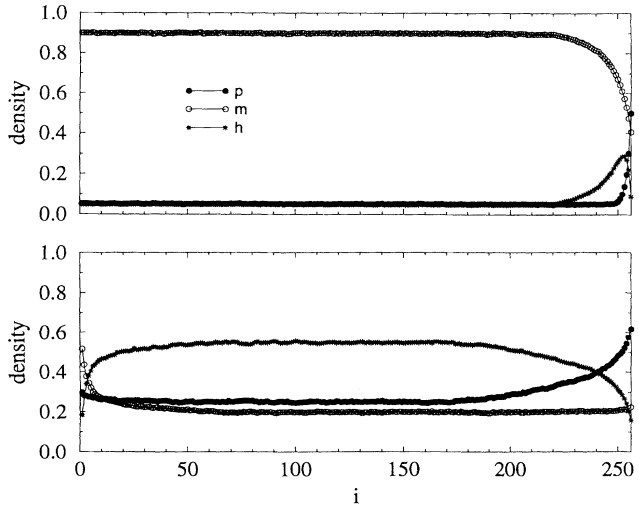

FIG. 2. Density profiles of the positive (p) and negative (m) charges and hole (h) in the hd-ld ($\alpha = 1, \beta = 0.1, q = 1$) and ld-ld ($\alpha = 1, \beta = 0.333, q = 1$) phases.

To examine this mechanism we consider small β, where the two coexisting states are basically either all positive ($\rho_+ \simeq 1$) or all negative ($\rho_- \simeq 1$). Suppose the system is in the $\rho_+ \simeq 1$ state. In this phase there is a low flux of negative charges and holes which enter the system at the right end and leave at the left end. As the system evolves, a blockage of negative particles (a droplet) may temporarily be formed at the left end due to some fluctuation. When this happens, holes enter the system from both ends resulting in a net influx of holes. They get trapped in between the positive and negative regions. A typical configuration would thus look like $(- - -0000+ + \cdots +)$. Usually, the block at the left end leaves the system after some time and the system relaxes to the all $(+)$ state. However, if the block persists for time of order N, the system will be filled with holes, and thus has a chance of switching to the negatively charged state. Therefore one is interested in the probability of a block persisting for a time of $O(N)$. To estimate this probability we consider the dynamics of the block. The right end of the block, located at a distance $x(t)$ from the left end of the system, performs a diffusive motion biased to the left. The bias is due to the fact that in the blocked state the influx of negative charges at the right end, $j_{\text{in}} = \beta\alpha/(1 + \alpha)$, is smaller than the flux leaving the system at the left end, $j_{\text{out}} = \beta$. The dynamics of $x(t)$ may thus be described by a biased random walk with absorption at $x = 0$. The probability $P(x_0, t)$ that such a walker starting at position x_0 reaches the origin at time t is given by

$$P(x_0, t) \propto \frac{x_0}{t^{3/2}} \exp[-(x_0 - v_t)^2/ct] \qquad (4)$$

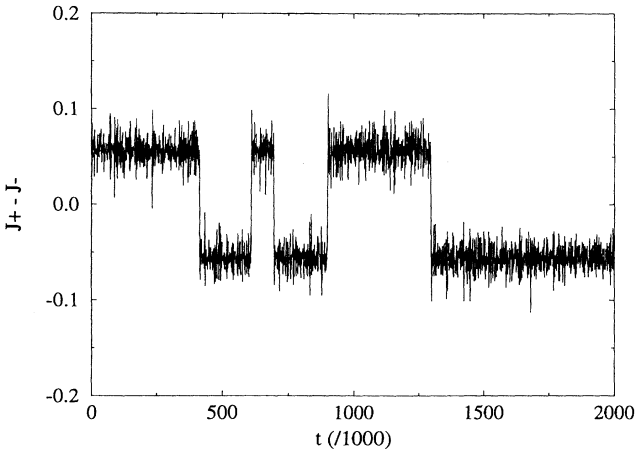

FIG. 3. Time evolution of the current difference in the hd-ld phase ($\alpha = q = 1, \beta = 0.15, N = 80$). Each point represents an average of the current difference over 1000 sweeps. Flips between the two symmetry related states are clearly seen.

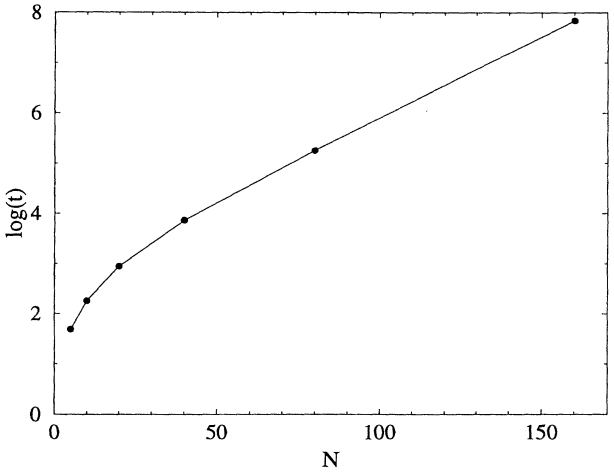

FIG. 4. Time scale $\tau(N)$ as a function of N for $N \leq 160$.

for large t. Here v is the average rate at which the droplet shrinks, and c is a constant. Within this picture the probability that a droplet of initial size of $O(1)$ persists for time of $O(N)$ is exponentially small, $\exp(-bN)$, where b is a constant. The time scale between flips is therefore of order $\exp(bN)$. A detailed discussion of the switching mechanism and the associated time scales will be given elsewhere [12].

This behavior is very different from that of a one species model on the coexistence line between the high and the low density states. In that case the two states have the *same* current and therefore a domain wall between the high density and the low density states is a *stable* object with a vanishing net velocity. It displays a diffusive motion and thus takes a time of order N^2 to traverse and hence flip the system.

To study the flipping process in a finite system we simulated the dynamics of Eqs. (1). In Fig. 3 we present the time evolution of the current difference $J_- - J_+$ for a typical run in the hd-ld broken symmetry phase. Similar behavior is found for the density difference between the two charges. It is clear from Fig. 3 that at any given time the two states do not coexist in the system (except, maybe, during the short time when a flip takes place). This supports the argument given above, namely that droplets of the "wrong" state are expelled from the system unless they are macroscopically large.

To evaluate the time scale $\tau(N)$ we averaged the current difference over many runs, starting from the initial configuration where all sites are occupied by positive charges. This average decays at large time t as $\exp[-t/\tau(N)]$ and thus yields $\tau(N)$. We have measured the time scale for systems of size 5, 10, 20, 40, 80, 160, and the results are given in Fig. 4. At first glance they seem to suggest that $\tau(N)$ grows somewhat slower than exponentially with N, maybe like $\exp(aN^\gamma)$ with $\gamma < 1$. However, in trying to fit the data to a stretched exponential form we find that γ tends to grow with the system size for $N \leq 160$. This may indicate that the stretched exponential form is related to finite size effects, and that, in fact, $\tau(N)$ grows exponentially for large N, as suggested by the droplet dynamics discussed above.

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