Nonlinear Resonant Absorption of Surface Magnetohydrodynamic Waves

J. Lundberg and G. Brodin
Department of Plasma Physics, Umeå University, S-901 87 Umeå, Sweden
(Received 2 May 1994)

We have derived two coupled equations that describe resonant absorption of surface magnetohydrodynamic (MHD) waves in the nonlinear regime. It turns out that the evolution of our system is drastically affected by the nonlinearity. The relevance of our results for the Alfvén wave heating scheme of fusion plasma as well as for the heating of the solar corona by surface MHD waves is discussed.

PACS numbers: 52.35.Mw, 52.35.Bj, 52.35.Nx, 52.50.Gj

Resonant absorption of surface magnetohydrodynamic (MHD) waves and the related issue of spectral properties of Alfvén waves in inhomogeneous plasmas have been studied by numerous authors; see, e.g., Refs. [1–4]. The works have been directed towards applications on heating of the solar corona [1,2], on heating of laboratory plasmas for controlled thermonuclear fusion research [3], or mainly concerned fundamental theoretical issues [4].

Almost all previous authors have considered the linear regime. However, Alfvén wave heating experiments have been performed which indicate that the driven Alfvén waves may undergo a nonlinear evolution [5]. This is not surprising, considering the field amplification that takes place at the resonance surface, where the local Alfvén wave frequency matches the surface wave frequency. In the present Letter we will thus study nonlinear resonant absorption of surface MHD waves. As we shall demonstrate, the nonlinearities change the physical picture dramatically. In contrast to the linear case, where the surface wave energy is irreversibly transferred to the resonant Alfvén waves, a large enough nonlinearity leads to oscillations of the energy between the surface wave and the resonant Alfvén waves.

For the sake of mathematical simplicity we have used the ideal incompressible MHD equations. As the plasmas of interest are compressible, this will limit the direct applicability of our results. However, our choice of basic equations is supported by the fact that compressibility effects, at least in the linear regime, are irrelevant around the resonance surface [6], where most of the interesting physics takes place. A discussion of relevant physical effects not described by our simple model will be given at the end of this Letter.

In order to study resonant absorption of surface MHD waves we assume that the background geometry is as follows: Inside a plasma slab, $|x| < x_1$, the unperturbed density, pressure, and magnetic field are given by ρ_i, p_i, and $\mathbf{B} = B_i\hat{\mathbf{z}}$, respectively. Outside the slab $|x| > x_2$, the corresponding values are ρ_e, p_e, and $\mathbf{B} = B_e\hat{\mathbf{z}}$. The two transition layer regions $x_1 < |x| < x_2$ are assumed to be thin, i.e., $\Delta x \equiv |x_2 - x_1| \ll x_1$. The background quantities must satisfy the pressure balance equation

$$p(x) + \frac{B^2(x)}{8\pi} = p_i + \frac{B_i^2}{8\pi} = p_e + \frac{B_e^2}{8\pi}, \qquad (1)$$

but may otherwise be taken to connect the unperturbed values in the inside and outside regions of the plasma slab in an arbitrary manner. For simplicity we restrict ourselves to wave propagation along the unperturbed magnetic field. We can then introduce the velocity and magnetic potentials $\mathbf{\Psi} = (0, -\Psi, 0)$ and $\mathbf{A} = (0, -A, 0)$, defined according to $\mathbf{v} = \nabla \times \mathbf{\Psi}$ and $\mathbf{B} = B(x)\hat{\mathbf{z}} + \nabla \times \mathbf{A}$. The equations of ideal incompressible MHD can then be written [7]

$$\rho\Psi_{zt} + P_x - \frac{1}{4\pi}B(x)A_{zz} = \frac{1}{4\pi}[A_zA_{xz} - A_xA_{zz}] + \rho[\Psi_x\Psi_{zz} - \Psi_z\Psi_{xz}], \qquad (2)$$

$$-\rho\Psi_{xt} + P_z - \frac{1}{4\pi}[B_x(x)A_z - B(x)A_{xz}] = \frac{1}{4\pi}[A_xA_{xz} - A_zA_{xx}] + \rho[\Psi_z\Psi_{xx} - \Psi_x\Psi_{xz}], \qquad (3)$$

and

$$A_t - B(x)\Psi_z = \Psi_xA_z - A_x\Psi_z, \qquad (4)$$

where P $(= p + B^2/8\pi)$ denotes the sum of thermal and magnetic pressures, and indices x, z, and t denote partial x, z, and t derivatives, respectively. The quadratic nonlinear terms will create low frequency as well as second harmonic perturbations. We thus make the ansatz $Q(x,z,t) = Q^{(0)}(x,t) + Q^{(1)}(x,t)\exp[i(kz - \omega t)] + Q^{(2)}(x,t)\exp[i(2kz - 2\omega t)] + \text{cc}$ where cc denotes complex conjugate and Q represents a particular oscillating quantity. The amplitudes are assumed to be slowly varying, i.e., $|(\partial Q^{(j)}/\partial t)/\omega Q^{(j)}| \ll 1$ for $j = 0,1,2$. We will also take the transition layer to be thin compared to the wavelength, $2\pi/k$, and consider the excursion length of the particles in the wave field to be much less than Δx, i.e., $|\Psi_x/\omega\Delta x| \ll 1$. By successive approximations making use of our small parameters, we find after lengthy but straightforward calculations

$$\varepsilon(x)\Psi + \frac{\omega}{k}P_x + 2i\omega\rho(x)\Psi_t + \frac{i}{k}P_{xt} = 0 \qquad (5)$$

and

$$\varepsilon(x)\Psi_x + k\omega P + 2i\omega\rho(x)\Psi_{xt} + ikP_t = -\frac{k^4}{4\pi\omega^2}[B^2(x)]_x\Psi^2\Psi_{xx}^*, \qquad (6)$$

 0031-9007/95/74(11)/1994(4)$06.00

where only the largest nonlinear term, that is due to second harmonic generation, has been kept. For notational convenience we have dropped the superscript (1) on all the amplitudes in Eqs. (5) and (6). When deducing Eqs. (5) and (6) we have assumed the linear scalings (and the lowest order perturbation expansion for $\Psi^{(2)}$, $\Psi^{(0)}$, etc.) to be valid as order of magnitude estimates [8], i.e., we have $\Psi_x \sim k\Psi(x_1)(e^{-ia(x)\omega t} - 1)/a(x)$, where $a(x) = [1 - k^2V_A^2(x)/\omega^2]/2$. This leads to the estimations $\Psi_x^{(2)} \sim k^2\Psi\Psi_x/\omega$, $\Psi_x^{(0)} \sim k^2\Psi\Psi_x^*/\omega$, $P \sim \Delta x\omega\rho(x)\Psi_x$, etc. around the resonance. The reason that nonlinear terms are included in Eq. (6) but not in Eq. (5) is that the magnitude of the ratio of nonlinear and linear terms in Eq. (5) are of higher order in a $k\Delta x$ expansion compared to that in Eq. (6). Note that we can use $\rho_0(x,t) = \rho_0(x,t=0) \equiv \rho(x)$ in Eqs. (5) and (6) as the nonlinear low frequency density modulations are comparatively weak. The function $\varepsilon(x)$ describes the coupling to the resonant Alfvén waves [9] and is given by

$$\varepsilon(x) = \rho(x)\,\omega^2\left(1 - \frac{k^2V_A^2(x)}{\omega^2}\right),$$

where V_A ($= B(x)/2[\pi\rho(x)]^{1/2}$) is the Alfvén velocity. Thus the local Alfvén wave frequency $kV_A(x)$ coincides with the surface wave frequency ω at the resonance surface $x = x_r$, defined by $\varepsilon(x_r) = 0$. As a consequence, the surface wave excites Alfvén waves around $x = x_r$, resulting in a strong field amplification of Ψ_x that must be taken into account when deducing Eq. (6).

We proceed by making the ansatz

$$\Psi = \Psi_2(t)\,e^{-k(|x|-x_2)}, \quad P = P_2(t)\,e^{-k(|x|-x_2)}, \tag{7}$$

for $x_2 \le |x|$, and take

$$\Psi = \Psi_I(t)\sinh(kx), \quad P = P_I(t)\cosh(kx), \tag{8}$$

inside the plasma slab $|x| \le x_1$. Thus we are using the same x dependences as for a *linear* slow sausage mode, which is natural as we are not interested in nonlinearities in the homogeneous regions. In order to find the relations between Ψ_2 and P_2, as well as between Ψ_I and P_I, we substitute the ansatz (7) and the ansatz (8) into Eqs. (5) and (6), where the nonlinearity can be neglected. Combining the resulting equations with the two trivial identities

$$\Psi_2 - \Psi_1 = \int_{x_1}^{x_2} \Psi_x\,dx \tag{9}$$

and

$$P_2 - P_1 = \int_{x_1}^{x_2} P_x\,dx\,, \tag{10}$$

where $\Psi_1 = \Psi(x_1)$ and $P_1 = P(x_1)$, we obtain the evolution equation

$$\frac{\partial P_1}{\partial t} = ia\int_{x_1}^{x_2} \Psi_x\,dx\,, \tag{11}$$

where

$$a = \frac{\cosh(kx_1)}{2\omega^2}\left[\frac{\rho_2\cosh(kx_1)}{\varepsilon_2^2} + \frac{\rho_1\sinh(kx_1)}{\varepsilon_1^2}\right]^{-1},$$

provided ω and k fulfill the dispersion relation

$$\varepsilon_2 + \varepsilon_1\coth(kx_1) + k\int_{x_1}^{x_2} \varepsilon(x)\,dx = 0\,. \tag{12}$$

Here $\varepsilon_1 = \varepsilon(\omega,k,x_1)$, $\varepsilon_2 = \varepsilon(\omega,k,x_2)$, and the last term is a correction of order $k\Delta x$ to the usual sharp boundary dispersion relation [10]. When deriving Eqs. (11) and (12) we have used that Ψ and P are approximately constant over the transition layer, i.e., that the right hand sides of Eqs. (9) and (10) are correction terms of order $k\Delta x$. Equation (11) describes the influence of the Alfvén wave field Ψ_x (corresponding to the velocity in the z direction) that essentially is localized around the resonance, on the surface wave field P_1, which determines the large scale structure outside the boundary. Next, we note that within the transition layer we can approximate $P(x,t)$ with $P_1(t)$ in Eq. (6), and use that the last term on the left hand side always is much smaller than the second term. Thus we obtain

$$2i\omega\rho(x)\Psi_{xt} + \varepsilon(x)\Psi_x + k\omega P_1 = -\frac{k^4}{4\pi\omega^2}[B^2(x)]_x\Psi^2\Psi_{xx}^*\,. \tag{13}$$

Equation (13) describes the evolution of the Alfvén wave field Ψ_x, which is driven by the surface wave field P_1. The two coupled equations (11) and (13) constitute one of the main results of this Letter. It is straightforward to generalize these equations to include the effects of propagation, i.e., z-dependent amplitudes; see Ref. [11]. In the linear approximation equations (11) and (13) give the same result as previous papers [4] studying resonant absorption of surface MHD waves, although, as far as we know, the form of our equations is new. A numerical study of the linearized version of Eqs. (11) and (13) thus gives the well known result of an irreversible energy flow from the surface wave field P_1 to the Alfvén wave field Ψ_x, resulting in a constant damping rate of the surface wave in the time-asymptotic limit.

Before we start a numerical investigation of the nonlinear system, it is instructive to consider the energy conservation properties. Multiplying Eq. (11) by P_1^*, adding the complex conjugate of the resulting equation, and applying Eq. (13) we obtain

$$\frac{\partial}{\partial t}\left[k\omega|P_1|^2 + 2a\omega\int_{x_1}^{x_2}\rho(x)|\Psi_x|^2\,dx\right] = 0\,. \tag{14}$$

When deriving Eq. (14) a term due to the nonlinearity has been omitted. Whenever the amplitude of the second harmonics is much smaller than the amplitude at the original frequency, which is a necessary condition for the validity of Eqs. (11) and (13), the contribution to Eq. (14) from the nonlinearity is indeed negligible, as is verified by our numerical calculations. All other amplitude restrictions we have used in the derivation are also seen to be fulfilled. Apart from a normalization, it is clear that the first term represents the energy of the surface wave, whereas the second term is the energy contribution from the Alfvén waves localized in the transition layer, i.e., the sum of Alfvén wave energy and surface wave energy is conserved.

Next, we shall solve the coupled equations (11) and (13) numerically. We use values of the unperturbed quantities that are appropriate for photospheric solar conditions. For a magnetic flux tube we can take $B_i = 1000$ G, $B_e = 10$ G, $x_1/\Delta x = 20$, $p_i = 0.62 \times 10^4$ dyn cm^{-2}, $p_e = 4.60 \times 10^4$ dyn cm^{-2}, and $\rho_e = 5.13 \times 10^{-8}$ g cm^{-3}, resulting in a total pressure $P_0 = 4.60 \times 10^4$ dyn cm^{-2}. The density ratio $\rho_e/\rho_i = 6$ yields $\rho_i = 0.855 \times 10^{-8}$ g cm^{-3}. If we take the half slab width to be $x_1 = 50$ km (5×10^6 cm) and $k = 1.6 \times 10^{-7}$ cm^{-1} ($\lambda = 3.93 \times 10^7$ cm), the frequency ω can be calculated from the linear dispersion relation (12) and its value is 0.219 s^{-1}. These data will be used throughout this Letter. The boundary conditions are that Ψ and Ψ_x are continuous at $x = x_1$. When generating the curves in Fig. 1, the initial conditions for $\Psi_x(x)$ are chosen according to $\Psi_x(x,0) = -k\omega P_1/[\varepsilon(x) + 2i\gamma\omega\rho(x_r)]$, where γ is the time-asymptotic linear damping [1]. This choice results in a constant damping rate for the surface wave in the linear regime, as it corresponds to an assumption that the Alfvén waves have initial phases such that they initially gain energy from the surface wave for all x values. The evolution of the surface wave field $P_1(t)$ is shown for various initial amplitudes $P_1(0)$. The dotted line corresponds to a completely linear evolution of $P_1(t)$. The other two curves correspond to various degrees of nonlinear behavior. We note that initially the energy transfer to the Alfvén waves is actually enhanced by the nonlinearity. To understand this we must observe how the $\Psi_x(x)$ profile is affected by the nonlinearity, which can be seen in Figs. 2(a) and 2(b). The linear profile [Fig. 2(a)] shows the characteristic oscillatory pattern resulting from phase mixing. The narrow peak centered around x_r corresponds to the region in which the Alfvén waves have gained energy from the surface wave on the average. The nonlinear profile [Fig. 2(b)] is not as sharply peaked as the linear one, i.e., for comparatively short times the effect of the nonlinearity is to create a broadened spectra of Alfvén waves that take energy from the surface wave. However, as we know, the linear damping for *long* times depends critically on the possibility for the resonant Alfvén waves at $x = x_r$ to gain energy indefinitely. As Ψ_x is highly peaked around the resonance, and the scale length for x variations tends to decrease due to phase mixing, which accordingly makes Ψ_{xx} very large, a fairly modest value of Ψ is enough to cause a nonlinear detuning of the resonant Alfvén waves and thus stop the continued absorption of surface wave energy. In agreement with this, the curve in Fig. 1, corresponding highly to nonlinear initial conditions, shows an oscillatory behavior after the strong damping occurring initially.

Now it turns out that the evolution of the system is rather sensitive to the initial conditions in the nonlinear regime. Thus, by choosing $\Psi_x(x,0) = -k\omega P_1 \exp[2\pi i(x - x_1)/\Delta x]/[\varepsilon(x) - 2i\gamma\omega\rho(x_r)]$, i.e., modifying the initial phase relation between the surface wave and the Alfvén waves, we find a significant change in the nonlinear behavior, as shown in Fig. 3. The dashed line corresponds to the new modified phase relation, and the solid line is the same as that shown in Fig. 1. Naturally we have used the same initial amplitude of the

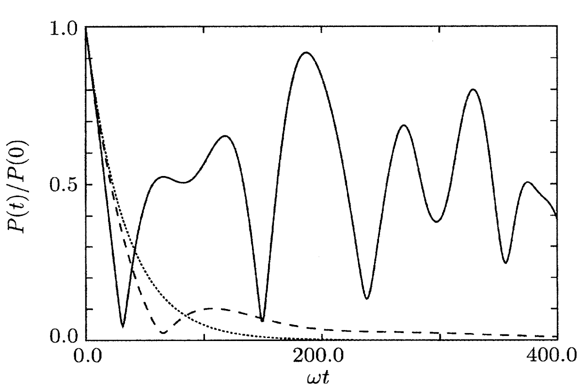

FIG. 1. The time evolution of the normalized wave field $P(t)/P(0)$ for various initial amplitudes $P_{\rm init} = P(0)/P_0$: $P_{\rm init} = 4.0 \times 10^{-3}$ (solid line), $P_{\rm init} = 1.33 \times 10^{-3}$ (dashed line), and $P_{\rm init} = 4.0 \times 10^{-8}$ (dotted line).

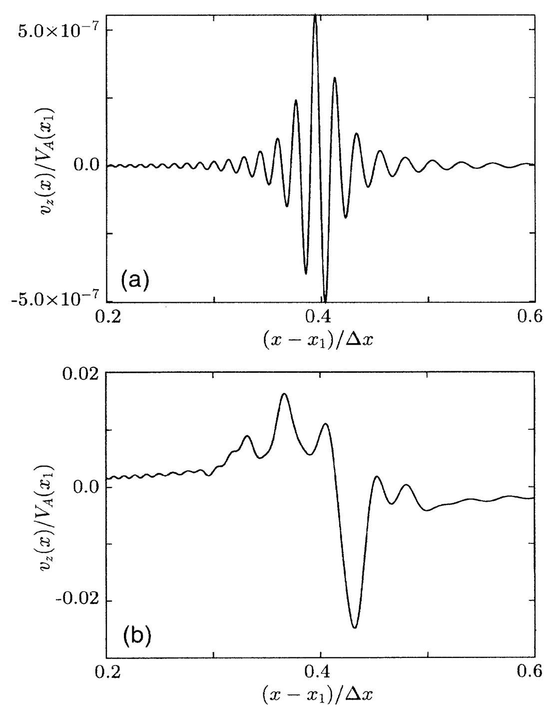

FIG. 2. $\mathrm{Re}[\Psi_x(x)]/V_A(x_1)$ versus x when $t = 200\omega^{-1}$. (a) The linear case, $P_{\rm init} = 4.0 \times 10^{-8}$. (b) The strongly nonlinear case, $P_{\rm init} = 4.0 \times 10^{-3}$.

surface wave in both cases. Apparently the oscillations of the surface wave amplitude are much less pronounced for the new initial conditions. A rough explanation of this somewhat different behavior can be given as follows: Details in the initial $\Psi_x(x)$ profile lead to energy gain and energy loss, respectively, for (fairly) neighboring oscillators, i.e., we get a more mixed Alfvén wave spectra for the new initial conditions. This means that the energy exchange with the surface wave becomes less coherent, and consequently the oscillations of the surface wave amplitude will be smaller. The two curves shown in Fig. 3 are together quite representative of the evolution of the surface wave amplitude in the nonlinear regime. Thus a general property of our system is that the irreversible energy transfer from the surface wave to the resonant Alfvén waves is replaced by oscillations of the energy in the nonlinear regime.

A natural question is how dependent our results are of the highly idealized model we have used. Essentially the properties of our system depend on three facts: (1) The existence of a linear Alfvén wave resonance; (2) conservation of the sum of Alfvén wave energy and surface wave energy; and (3) a nonlinear detuning of the Alfvén wave resonance.

If compressibility is introduced, the single resonance of our Letter splits into two resonances, the so called slow resonance [2] and the Alfvén wave resonance. However, the energy corresponding to the slow mode also propagates only along the magnetic field. Hence the same physics as described above is expected for the compressible case. The introduction of finite Larmor radius effects destroys point (2), however. Because of these effects, the Alfvén waves can propagate away from the resonance, perpendicular to the magnetic field towards the side of lower ε, and become Landau damped at some distance from the resonant surface. Whether nonlinear effects will still be of importance depends on the ratio between the excursion length of the particles in the Alfvén wave field and the average Larmor radius of the ions, i.e., a large enough value of this ratio means that finite Larmor radius effects can be neglected. The main effect of a finite resistivity is on the linear Alfvén resonance (the effect on the nonlinear terms is much less important). Naturally this inclusion of a dissipation mechanism destroys point (2), although such an effect usually is of minor importance compared to the finite Larmor radius effects [6].

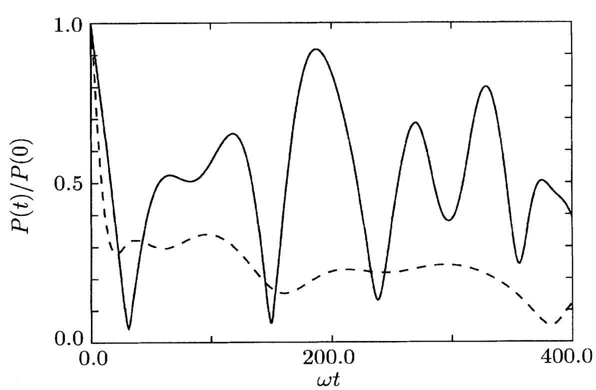

FIG. 3. The time evolution of the normalized wave field $P(t)/P(0)$ in the strongly nonlinear regime where $P_{\text{init}} = 4.0 \times 10^{-3}$. The two curves correspond to different initial phase relations between the surface wave and the Alfvén waves.

Resonant absoprtion is involved in the Alfvén wave heating scheme of fusion plasmas [3], as well as in a popular theory for the heating of the solar corona [1,2]. Our results suggests that nonlinear effects can be of importance in these contexts, i.e., that for large surface wave amplitudes the energy transfer to the resonant Alfvén waves can be obstructed due to nonlinear effects. In order to make a fully realistic estimate of what the influence of the nonlinearity may be, more work needs to be done, however. For example, it is not clear from our treatment whether the nonlinearities will act to reduce or increase the absorption rate when finite Larmor radius effects are comparatively large, as is the case in many applications. The former alternative seems to be the more probable, however. Furthermore, the external driving currents should be included in the fusion application, as well as the effects of the toroidal geometry.

[1] A.J. Ionson, Astrophys. J. **226**, 650 (1978); T. Sakurai, M. Goossens, and J.V. Hollweg, Solar Phys. **133**, 247 (1991); M. Goossens, J.V. Hollweg, and T. Sakurai, Solar Phys. **138**, 233 (1992).
[2] T. Sakurai, M. Goossens, and J.V. Hollweg, Solar Phys. **133**, 227 (1991).
[3] A. Hasegawa and L. Chen, Phys. Rev. Lett. **35**, 370 (1975); A. Hasegawa and L. Chen, Phys. Rev. Lett. **36**, 1362 (1976); S. Poedts, W. Kerner, and M. Goossens, J. Plasma Phys. **42**, 27 (1989); J. Vaclavik and K. Appert, Nucl. Fusion **31**, 1945 (1991).
[4] C. Uberoi, Phys. Fluids **15**, 1673 (1972); J.A. Tataronis, J. Plasma Phys. **13**, 87 (1975); M.A. Lee and B. Roberts, Astrophys. J. **301**, 430 (1986); C. Zorzan and P.S. Cally, J. Plasma Phys. **47**, 321 (1992).
[5] G.G. Borg, J.B. Lister, S. Dalla Piazza, and Y. Martin, Nucl. Fusion **33**, 831 (1993).
[6] A. Hasegawa and C. Uberoi, *The Alfvén Wave* (U.S. DOE, Oak Ridge, 1985).
[7] M.S. Ruderman, J. Plasma Phys. **49**, 271 (1993).
[8] For the most nonlinear initial conditions we have used, the linear scaling for Ψ_x is not obeyed. Inclusion of additional terms in our numerical code shows that Eq. (6) is still an excellent approximation, however.
[9] In the compressible case there are actually two resonances, usually called the Alfvén resonance and the slow resonance (or cusp resonance). The two resonances coincide in the incompressible limit.
[10] B. Roberts, in *Cometary and Solar Plasma Physics*, edited by B. Buti (World Scientific, Singapore, 1988).
[11] G. Brodin and J. Lundberg, Phys. Plasmas **1**, 96 (1994).