Superinstantons in Gauge Theories and Troubles with Perturbation Theory

Adrian Patrascioiu* and Erhard Seiler
Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Föhringer Ring 6, 80805 Munich, Germany
(Received 8 February 1994)

In gauge theories with continuous groups there exist classical solutions whose energy vanishes in the thermodynamic limit (in any dimension). The existence of these *superinstantons* is intimately related to the fact that even at short distances perturbation theory can fail to produce unique results. This problem arises only in non-Abelian models and only starting at $O(1/\beta^2)$, and is expected to modify the universal coefficients of the β function.

PACS numbers: 11.15.Bt, 11.15.Ha

The standard belief in quantum field theory is that in asymptotically free theories perturbation theory predicts correctly the short distance behavior of the Green's functions—and that the harder part is understanding their infrared behavior. In this Letter we will show via some simple calculations that this belief is erroneous: In these theories the ultraviolet (uv) and infrared (ir) effects are entangled in such a way that, in fact, perturbation theory fails even at short distances; in particular, the main assumption underlying the applications of the operator product expansion (regarding a certain factorization of uv and ir effects) is invalid in theories such as QCD.

Since our claim runs against the established beliefs, we would like to provide a clear exposition of the issues involved. Consequently, at the risk of boring some readers, we will repeat the general discussion needed to comprehend what is the trouble [1]. We begin by reminding the reader why one needs a nonperturbative definition of a quantum field theory. First, if one claims, as is usually done, that perturbative QCD (PQCD) is correct at short distances, what exactly is one claiming? The temptation is usually to appeal to experiment, but that is silly since most gauge models have nothing to do with nature and one could still ask the question. The only way to make sense of such a claim is to give the theory a nonperturbative definition and to argue that in a certain regime perturbation theory produces a good approximation.

A second reason underlying the need for a nonperturbative definition of a quantum field theory comes from the fact that, correct or not, perturbation theory provides us with answers in the form of divergent (nonconvergent) series. Whereas a unique numerical answer can be associated with any convergent series, divergent power series have no intrinsic meaning, but become meaningful only if some external (nonperturbative) definition is provided. Currently popular attempts to "sum up" perturbation theory (PT) are therefore meaningless in the absence of a nonperturbative definition of the theory.

Therefore both to understand the meaning of the question "is PQCD correct at short distances?" and to make sense of the predictions of PQCD, one needs a nonperturbative definition of QCD. We will adopt the lattice (LQCD) approach. Some readers may conclude that our unpleasant conclusions are a consequence of our use of LQCD. Although that is logically possible, unless an alternative nonperturbative framework is proposed and the troubles appearing in LQCD are shown to disappear, such an excuse cannot be taken seriously.

Following Wilson [2], we consider a regular cubic lattice in D dimensions and define the partition function as

$$Z_\Lambda = \int \prod_{x,\mu} dU_\mu(x) e^{\beta/N \Sigma_{\rm plaq} {\rm tr}\, U_{\rm plaq}}. \tag{1}$$

Here $U \in \mathrm{SU}(N)$ or $U \in \mathrm{U}(1)$ and $\Lambda \subset \mathbf{Z}^D$ is a hypercube of linear extension L. To fully specify the problem we must impose some boundary conditions (BC). We will choose the following BC: in the hyperplane $x_1 = 0$, all $U_{\rm plaq} = 1$, while for all links lying in the hyperplane $x_1 = L$ $U_{\rm link}$ are free variables; in the remaining directions Λ is a hypertorus (periodic BC) with no restriction on the Polyakov loops in these directions (see Fig. 1).

We chose these BC because they are easily compatible with the maximally axial gauge. For simplicity we will discuss the case $D = 3$, the extension to higher D being obvious. For our BC the maximally axial gauge amounts to $U_1(x) = 1$, $x \in \Lambda$, and $U_2(0,i,j) = 1$ for all (i,j). Fixing the gauge is necessary if we wish to do PT. Indeed, as $\beta \to \infty$, once the gauge has been fixed and the BC specified, the system will perform small oscillations

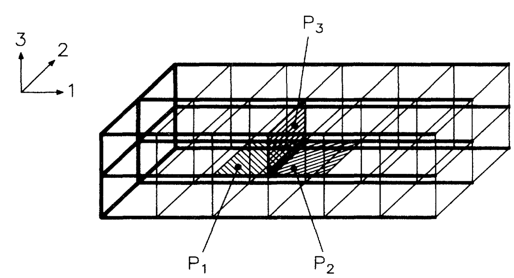

FIG. 1. The lattice in 3D. On the links drawn in heavy lines $U_{\rm link} = 1$. On the link $(m,2)$ common to the plaquettes P_1, P_2, and P_3, $U_2(m) = 1$ for SI BC.

 0031-9007/95/74(11)/1924(4)$06.00

around one or several (classical) configurations. PT is simply the saddle point approximation built around these classical configurations. For L finite PT is correct in the following sense. Let $G(\beta)$ be the expectation value of some gauge invariant observable. Then for $\beta \to \infty$ PT produces the correct asymptotic expansion of $G(\beta)$ in powers of $1/\beta$. The mathematical meaning of the statement is that if one truncates the PT formal series at order k, the error is $o(1/\beta^k)$.

Therefore to verify the correctness of a certain PT prediction one needs not only a computation of the coefficients entering the expansion of $G(\beta)$, but also a bound on the truncation error. That is easy to do for finite L. Unfortunately, for non-Abelian groups the best estimates of the truncation error are such that they diverge as $L \to \infty$ (for Abelian groups a bound uniform in L has been shown to exist [3]). Thus it is mathematically unknown if the PT expansion of a short distance quantity, such as the plaquette energy, remains valid as $L \to \infty$.

Since many things which are true are nevertheless hard to prove, one may wonder if there is any cause for doubting PT. As emphasized several years ago by Patrascioiu [4], there are reasons to do so. The saddle point approximation of an integral amounts to replacing the integrand by a Gaussian plus corrections. Intuitively one would guess that such an approximation is good provided the integrand is sharply peaked and the peak is sufficiently far from the boundaries of the integration region. Now in the maximally axial gauge, Patrascioiu showed that, as $L \to \infty$, the integrand becomes arbitrarily flat; that is, in that gauge, for any D, no matter how large β is, for L sufficiently large one will encounter large fluctuations. This statement is much stronger than Elitzur's theorem [5]: It says that in gauge theories, for any D, in the maximally axial gauge there is no long-range order for any finite β. Since PT is an expansion in small deviations from an ordered state and such a state does not exist on an infinite lattice, Patrascioiu concluded that even at short distances PT was highly suspicious.

One may wonder if Patrascioiu's conclusion could not be dismissed as a gauge artifact, since it is well known that at least for $D > 2$ the ir divergences which PT displays in this gauge disappear in other gauges. Unfortunately, the answer is that this is not a gauge choice artifact, but a genuine difficulty of gauge theories. To see that consider first the fact that the asymptotic expansion of $G(\beta)$, if it exists, is unique. Now for finite L the maximally axial gauge is definitely usable. Whatever answers PT produces in this gauge they must represent the true asymptotic expansion of $G(\beta)$ and any other gauge choice is bound to reproduce them. Since the PT answers in any other gauge must agree with the PT answers in the maximally axial gauge for finite L, they must also agree for $L \to \infty$. Therefore if PT is not uniformly valid for $L \to \infty$, that fact has nothing to do with the use of the maximally axial gauge.

A direct corroboration of this conclusion comes from the fact that, while trying to avoid the large-fluctuations problem of the maximally axial gauge by using the Landau gauge, Zwanziger [6] discovered a new trouble which makes PT suspicious: as $L \to \infty$, the boundary of the integration region (the "fundamental modular domain") collides with the position of the peak of the integrand.

To recap the discussion presented so far, there are good reasons to suspect that in gauge theories, for any D, taking the termwise $L \to \infty$ limit of the PT coefficients may actually produce incorrect answers in non-Abelian models. To go beyond this point we would need an actual estimate of the truncation error, a task beyond our present abilities. Following a similar ruse to that we used recently in the 2D nonlinear σ models [7], we will avoid that task by appealing to a special property which these models have and which a correct PT computation should also exhibit. That property is the absence of symmetry breaking, which in the 2D σ models is the Mermin-Wagner theorem; in gauge theories in the maximally axial gauge it has been proven by Lüscher [8] as well as Simon and Yaffe [9] and also follows from Patrascioiu's observations [4]. This property implies that, in the infinite-volume limit, in the maximally axial gauge the expectation value of the energy of a plaquette located in the middle of the lattice is the same, whether or not we fix an additional link variable $U_2(m)$ at some link $(m,2)$ to some random value. The correctness of this statement is easy to verify via the convergent strong coupling cluster expansion [10]. We also performed a Monte Carlo study and verified numerically for SU(2) at $\beta = 3.0$ that the expectation values of the three plaquettes sharing the link $(m,2)$ and shown in Fig. 1 with $U_2(m) = 1$ converge to the same value, representing the thermodynamic value of this observable.

Next we would like to inquire whether the PT answers respect this property of the model. For that purpose we must compare the limit $L \to \infty$ in two PT computations: the first one with what we shall call the Dirichlet BC, described below Eq. (1), and the second one with what we shall call superinstanton (SI) BC, namely Dirichlet plus $U_2(m) = 1$. For the Dirichlet case the algebra has been carried out to a large extent by Müller and Rühl [11], whose procedure we followed. It yields the following infinite volume expression for the PT expansion of the plaquette energy:

$$\langle E \rangle \equiv \langle \mathrm{tr} U_{\mathrm{plaq}} \rangle = 1 - c_1/\beta - c_2/\beta^2 + \cdots . \qquad (2)$$

For the gauge group U(1) the values of the coefficients are

$$c_1 = 1/3, \qquad c_2 = 1/18, \qquad (3)$$

whereas for SU(2) they are

$$c_1 = 1, \qquad c_2 \approx 0.23. \qquad (4)$$

We have compared the SU(2) value of the c_2 obtained by us in this maximally axial gauge with that obtained by

Wohlert, Weisz, and Wetzel [12] in covariant gauges and they agree.

To obtain the PT coefficients with the SI BC we need a modified propagator, which vanishes on the link $(2, m)$. Following a suggestion made to us by Sokal (private communication), given a certain propagator $G(x, y)$ the combination

$$\tilde{G}(x,y) = G(x,y) - G(x,0)G(0,y)/G(0,0) \quad (5)$$

will be the propagator with the additional BC that it vanishes at 0. Therefore out of the previous (Dirichlet) propagators one can easily construct the SI propagators and thus compute the PT coefficients for these BC. We computed the coefficients c_1 and c_2 and found that $\lim_{L\to\infty} c_1(L) = 1$ even with SI BC. Our results of the computation of $c_1(L)$ with SI BC, which are identical for U(1) and SU(2) up to the trivial factor 3, are given in Table I; the results for $c_2(L)$ with SI BC are given in Tables II(a) and II(b), for the gauge groups U(1) and SU(2), respectively. For the U(1) model it can be seen that both $c_1(L)$ and $c_2(L)$ are approaching the values given in Eq. (3); it is easy to describe the values by a fit to a third-degree polynomial in $1/L$, with the constant term fixed to the values given in Eq. (3). The remarkable finding is that for the SU(2) model the two plaquettes P_1 and P_2 (see Fig. 1) sharing the same frozen link $U_2(m) = 1$ have PT coefficients c_2 converging to different values—and, in fact, only the limit of the coefficient of P_2 agrees with that obtained with Dirichlet BC. The data for P_1, however, are perfectly described by

$$c_2(L) = 0.40633 - 1.16026/L - 0.49819/L^2 \quad (6)$$

so the $\lim_{L\to\infty} c_2(L)$ is clearly different from the number 0.23 obtained with Dirichlet BC. As the discussion of the vacuum structure below shows, the mechanism responsible for this effect operates in any dimension $D \geq 2$. Therefore, as stated in the introduction, in non-Abelian models PT fails to reproduce the true properties of the model, such as the independence of the expectation value of the energy upon the BC used to reach the thermodynamic limit. This effect occurs only at $O(1/\beta^2)$ because only from that order on PT computations involve loop integrations which mix low and high momenta. Also the effect does not occur in Abelian theories, which contain no canceling ir divergences (the action depends only on gradients and the link measure is flat). Finally, let us notice that a similar effect should appear in a plaquette-plaquette two-point function from which one could determine the Callan-Symanzik β function. Thus the PT computation of the latter probably suffers from the same ambiguities and is not universal, as usually claimed. We verified that this actually happens in the 2D nonlinear σ model [7].

A few years ago Gribov [13] suggested that the long-distance behavior of QCD_4 is different from the usual picture of "infrared slavery," but that nevertheless PT describes correctly its short distance behavior. Our present results show, however, that even at short distances PT should not be trusted. On the other hand, what we found here lends support to the scenario presented by us [14], according to which $LQCD_4$ undergoes a zero temperature deconfining transition at finite β; such a transition would lead to a slower variation of α_s with the energy than predicted by PQCD—a prediction which has since found some experimental support from the LEP data—and hence to a different β function.

Some readers may wonder if there is a connection between the troubles with PT we have been pointing out and the claims [15] that β is a "bad expansion parameter" and needs to be replaced by an "improved one." The answer is no: the question we are addressing is not whether β is a good expansion parameter, but rather whether conventional PT gives the correct asymptotic expansion.

Also some readers may wonder if our results are not contradicted by the constructions of continuum Yang-Mills theory by Magnen, Rivasseau, and Sénéor [16], which also claim to establish asymptotic freedom. These constructions work in a small volume, precisely to avoid the large infrared fluctuations which are responsible for the effects we are describing in this paper.

Next let us explain the connection between these troubles of PT and the structure of the vacuum. In the maximally axial gauge the trouble arises because as L grows the system becomes less and less ordered. How

TABLE I. The PT coefficients $c_1(L)$ for the energies of the plaquettes P_1 and P_2 computed with superinstanton BC in the SU(2) model. For the U(1) model these numbers have to be divided by 3.

L	8	10	12	16	20	30
P_1	0.7587	0.8051	0.8364	0.8762	0.9004	0.9331
P_2	0.9967	0.9968	0.9971	0.9978	0.9981	0.9987

TABLE II. The PT coefficients $c_2(L)$ for the energies of the plaquettes P_1 and P_2 computed with superinstanton BC. (a) U(1) model; the data in the two rows are perfectly described by $1/18 - 0.15579/L + 0.19431/L^2 - 0.39615/L^3$ (P_1) and $1/18 - 0.005972/L + 0.01734/L^2$ (P_2). (b) SU(2) model; the data in two rows are perfectly described by $0.40633 - 1.16026/L - 0.49819/L^2$ (P_1) and $0.23522 + 0.12485/L - 5.84899/L^2$ (P_2).

L	8	10	12	16	20	30
			(a)			
P_1	0.03834	0.04152	0.04369	0.04648	0.04820	0.05056
P_2	0.05508	0.05513	0.05518	0.05525	0.05530	0.05538
			(b)			
P_1	0.2536	0.2852	0.3061	0.3320	0.3472	0.3669
P_2	0.1594	0.1893	0.2050	0.2201	0.2268	0.2329

does that happen? To understand that let us consider the energy of the classical configuration obtained by using Dirichlet BC (in the maximally axial gauge) and fixing $U_2(L/2,0,0) = V$ for some $V = \exp(i\frac{\vec{\tau}}{2}\cdot\vec{C}) \in$ SU(2). We cannot write the analytic expression for this classical configuration, but its general features are easy to comprehend. To that end let us write

$$U_\mu(x) = \exp\left[i\frac{\vec{\tau}}{2}\cdot\vec{A}_\mu(x)\right]. \tag{7}$$

The BC require $\vec{A}_2(0,0,0) = 0$ and $\vec{A}_2 = (L/2,0,0) = \vec{C}$. Consistent with these BC, suppose that

$$\vec{A}_2(x_1,0,0) = \frac{x_1}{L}\vec{C}. \tag{8}$$

The total energy of the (1,2) plaquettes lying between $x_1 = 0$ and $x_1 = L/2$ and having $x_2 = 0 = x_3$ is $O(C^2/L)$, hence it vanishes as $L \longrightarrow \infty$. Of course other plaquettes will also carry energy, but the total energy of the configuration will nevertheless vanish as $1/L$ for any D (we verified this numerically for $D = 3$). Indeed a gauge invariant description of the configuration we are discussing is this: there is a thin Wilson loop of length $L/2$ (width 1 lattice spacing) having the value $\exp(i\frac{\vec{\tau}}{2}\cdot\vec{C})$. The magnetic field is $O(|\vec{C}|/L)$ and falls off as r^{-D} as one goes transversely away from the loop. Consequently, even though in this configuration $A_\mu = O(1)$, its energy vanishes as $L \longrightarrow \infty$.

This is why we called these classical configurations superinstantons. Since they have arbitrarily low energy (and a lot of entropy) they will occur copiously in any gauge theory at weak coupling. In fact, the typical configuration at weak coupling could be regarded as a gas or liquid of superinstantons. This picture differs from the so-called "spaghetti vacuum" [17], which has higher free energy. In the 2D $O(N)$ σ models certain percolation results [7] allow one to conclude that if the typical configuration looks like a gas of superinstantons, then the model must be massless. In gauge theories such a connection is missing so far.

Nevertheless, since the superinstantons are practically degenerate with the trivial vacuum $\vec{A} = \vec{0}$, and since they are classical solutions, any saddle point expansion ought to incorporate them. How one would do that is unclear at the present time (would a dilute gas treatment be legitimate?). However, let us point out the following fact, related to the computation presented in this paper: One could consider PT around a given superinstanton (the calculations presented before correspond to PT around a superinstanton with $\vec{C} = \vec{0}$). In non-Abelian models, one would expect the result to depend on $\vec{C}$, since in the maximally axial gauge the ir divergencies are $O(L)$, whereas the new vertices induced by the superinstanton field are $O(1/L)$. These considerations show that in non-Abelian models the coefficient of the correct asymptotic expansion in the infinite volume limit at $O(1/\beta^2)$ may be different from what has been accepted as the truth so far. On the contrary, no such problem arises in the Abelian model since there are no canceling ir divergencies; hence PT around any superinstanton will reproduce PT around the trivial ground state, which makes understandable the result of [3] that PT produces the correct infinite volume asymptotic expansion.

A. P. is grateful to the Alexander von Humboldt Foundation for a Senior U.S. Scientist Award and to W. Zimmermann for his hospitality at the Max-Planck-Institute. We thank P. Weisz for information regarding PT in covariant gauges.

*Permanent address: Physics Department, University of Arizona, Tuscon, AZ 85721.

[1] A. Patrascioiu and E. Seiler, Report No. MPI-Ph/91-88.
[2] K. Wilson, Phys. Rev. D **10**, 2445 (1974).
[3] J. Bricmont, J.-R. Fontaine, J.L. Lebowitz, E.H. Lieb, and T. Spencer, Commun. Math. Phys. **78**, 545 (1981).
[4] A. Patrascioiu, Phys. Rev. Lett. **54**, 2292 (1985).
[5] S. Elitzur, Phys. Rev. D **12**, 501 (1975).
[6] D. Zwanziger, Nucl. Phys. **B378**, 525 (1992).
[7] A. Patrascioiu and E. Seiler, preceding Letter, Phys. Rev. Lett. **74**, 1920 (1995).
[8] M. Lüscher, Report No. DESY-77/16.
[9] B. Simon and L.G. Yaffe, Phys. Lett. **115B**, 145 (1982).
[10] K. Osterwalder and E. Seiler, Ann. Phys. (N.Y.) **110**, 440 (1978).
[11] V.F. Müller and W. Rühl, Ann. Phys. (N.Y.) **133**, 240 (1981).
[12] R. Wohlert, P. Weisz, and W. Wetzel, Nucl. Phys. **B259**, 85 (1985).
[13] V.N. Gribov, Phys. Scr. **T15**, 164 (1987).
[14] A. Patrascioiu and E. Seiler, in *Rencontre de Physique de la Vall'ee d'Aoste,* edited by M. Greco (Frontières, Gif sur Yvette, France, 1992).
[15] G.P. Lepage and P.B. Mackenzie, Phys. Rev. D **48**, 2250 (1993).
[16] J. Magnen, V. Rivasseau, and R. Sénéor, Commun. Math. Phys. **155**, 325 (1993).
[17] H.B. Neilsen and P. Olesen, Nucl. Phys. **B160**, 380 (1979).