A Local View of the Observable Universe

Marco Bruni,[1,2,3] Sabino Matarrese,[4] and Ornella Pantano[4]
[1]*Astronomy Unit, School of Mathematical Sciences, Queen Mary & Westfield College, Mile End Road, E1 4NS London, United Kingdom*
[2]*Dipartimento di Astronomia, Università di Trieste, via Tiepolo 11, 34131 Trieste, Italy*
[3]*Scuola Internazionale Superiore di Studi Avanzati, via Beirut 2-4, 34013 Trieste, Italy*
[4]*Dipartimento di Fisica "Galileo Galilei," Università di Padova, via Marzolo 8, 35131 Padova, Italy*
(Received 15 July 1994)

We present results on the nonlinear dynamics of inhomogeneous cosmological models with irrotational dust and a positive cosmological constant, considering, in particular, a wide class with vanishing magnetic Weyl tensor. We find that de Sitter is the unique attractor for those patches of the Universe that are able to expand (cosmic no-hair theorem). For the recollapsing regions we find a family of (Kasner) attractors, so that generically these regions fall in spindlelike singularities. These results give substantial support to the idea that the Universe can be very inhomogeneous on ultralarge, superhorizon scales, with observers living in those (almost) isotropic regions that emerge from an inflationary phase.

PACS numbers: 98.80.Hw, 04.40.Nr, 98.80.Cq

Cosmology in the 20th century has been essentially based on the 2 (+1) parameter (H_0, Ω_0, and q_0) standard Friedman-Robertson-Walker (FRW) models. In 1981 inflation came on the scene as a possible solution to the conundrums of the big-bang FRW scenario [1,2]. However, despite the fact that these flaws are in one way or another related to the question FRW models cannot answer, *why the observable Universe looks isotropic,* and the inflationary scenario was proposed also to answer this question, in practice most of the work on inflation has been done in the framework of isotropic FRW models. It was indeed soon recognized that during an inflationary phase initially present *small perturbations* are swept away, a fact that led to the conjecture that this could be the signature for a more general property of inflation, going under the name of cosmic no-hair theorem [3]. Roughly speaking, inflation should erase previously present inhomogeneities, leaving us with a unique possible observable universe: the isotropic one. Investigations to prove some restricted version of this general conjecture were done first in the framework of homogeneous anisotropic models [4], then also considering inhomogeneous spacetimes [5], but with practical examples limited to geometries with some degree of symmetry (e.g., [2,6], and references therein). The dominant perspective emerging from this analysis is that in a universe model either inflation occurs everywhere or there is no inflation at all. Instead, with truly inhomogeneous initial conditions, one can expect that there will always be patches of the Universe that will inflate and isotropize, while others will not, eventually recollapsing [7]. In this perspective, we propose that a weak cosmic no-hair theorem holds.

It is the aim of this Letter [8] to present some results about the evolution of inhomogeneous universes with irrotational dust of density ρ and a positive cosmological constant Λ, in particular, studying the local dynamics of the wide class of models with a vanishing magnetic part of the Weyl tensor, $H_{ab} = 0$.

These spacetimes were first considered in [9], while their first cosmological implementation was given in [10], where it was also shown that for the case of dust their time evolution is given by a system of six first-order ordinary differential equations for ρ (matter density), Θ (expansion scalar), σ_1 and σ_2 (two independent eigenvalues of the traceless shear tensor σ_{ab}), and E_1 and E_2 (eigenvalues of the traceless electric Weyl tensor E_{ab}). Thus, the evolution of each "fluid element" in the $H_{ab} = p = 0$ models is purely local once initial conditions satisfying the appropriate constraint equations [9,11] are given. Because of this property, these models were dubbed silent universes [12,13]. As shown in [12], $H_{ab} = 0$ is a good approximation (at least at second order in perturbations of a FRW background) outside the Hubble horizon, where also pressure gradients can be neglected, so that these models may provide a fair picture of the Universe on ultralarge scales. In other words, the time evolution of each superhorizon sized "volume element" of the universe should be well described by the equations of the $H_{ab} = 0$ models.

The dynamics of these models with $\Lambda = 0$ has been presented in detail elsewhere [13]. Here we only point out that they put the flatness problem [1,2,14] in a rather unusual perspective: Indeed, independently of its initial value, Ω ultimately tends to zero, both for expansion and collapse, with most of the universe volume dominated by expanding voids. Defining $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \sigma_2)$ and $E_\pm = \frac{1}{2}(E_1 \pm E_2)$ the dynamics of irrotational dust with $H_{ab} = 0$ and $\Lambda > 0$ is given by

$$\dot{\rho} = -\Theta\rho\,, \tag{1a}$$

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - 2\sigma^2 - \tfrac{1}{2}\rho + \Lambda\,, \tag{1b}$$

 0031-9007/95/74(11)/1916(4)$06.00

$$\dot{\sigma}_+ = \sigma_+^2 - \tfrac{1}{3}\sigma_-^2 - \tfrac{2}{3}\Theta\sigma_+ - E_+ , \quad (1c)$$

$$\dot{\sigma}_- = -2\sigma_+\sigma_- - \tfrac{2}{3}\Theta\sigma_- - E_- , \quad (1d)$$

$$\dot{E}_+ = \sigma_- E_- - 3E_+\sigma_+ - \Theta E_+ - \tfrac{1}{2}\rho\sigma_+ , \quad (1e)$$

$$\dot{E}_- = 3\sigma_- E_+ + 3E_-\sigma_+ - \Theta E_- - \tfrac{1}{2}\rho\sigma_- , \quad (1f)$$

where $\sigma^2 = 3\sigma_+^2 + \sigma_-^2$ is the shear magnitude. Setting $\sigma_- = E_- = 0$ one obtains the dynamics of a set of universe models generalizing those of Szekeres to the case of $\Lambda > 0$; some of these models were explicitly given in [15,16]. Equations (1a) and (1b) are actually more general than for the case $H_{ab} = 0$, as they hold for irrotational dust in general. Also, in this case the curvature of the 3-surfaces orthogonal to the matter 4-velocity [17] u^a is

$$^{(3)}R = -\tfrac{2}{3}\Theta^2 + 2\sigma^2 + 2\rho + 2\Lambda . \quad (2)$$

Proofs of cosmic no-hair theorems for homogeneous [4] and inhomogeneous [5] spacetimes were essentially based on the assumption $^{(3)}R \le 0$, although inflation has been found also in some $^{(3)}R > 0$ models [2,6,18]. For $^{(3)}R < 0$, it follows that $\Theta \ge \sqrt{3\Lambda}$, while $\dot{\Theta} \le \Lambda - \frac{1}{3}\Theta^2 \le 0$, so that if at t_* (an arbitrary initial time) $\Theta_* > 0$ (the model is initially expanding), then the model expands forever, with Θ squeezed between the lower and upper bound and $\Theta \to \sqrt{3\Lambda}$ exponentially, thus approaching (at least locally) a de Sitter spacetime in a time scale $\alpha = \sqrt{3/\Lambda}$.

Let us now drop the assumption $^{(3)}R \le 0$ and make some general (i.e., not restricted to the $H_{ab} = 0$ case) remarks. From (1b) and $\Theta_* > \sqrt{3\Lambda}$ the upper bound

$$\frac{\Theta(t,\vec{x})}{\sqrt{3\Lambda}} \le \coth\left[\sqrt{\frac{\Lambda}{3}}(t - t_*) + \operatorname{arccoth}\left(\frac{\Theta_*(\vec{x})}{\sqrt{3\Lambda}}\right)\right] \quad (3)$$

still holds, as in [4,5], but for $\Theta_* < \sqrt{3\Lambda}$ this becomes

$$\frac{\Theta(t,\vec{x})}{\sqrt{3\Lambda}} \le \tanh\left[\sqrt{\frac{\Lambda}{3}}(t - t_*) + \operatorname{arctanh}\left(\frac{\Theta_*(\vec{x})}{\sqrt{3\Lambda}}\right)\right] . \quad (4)$$

Defining as usual [13,19] by $3\dot{\ell}/\ell = \Theta$ the local scale factor ℓ, from (1a) one has that $\rho \to 0$ as long as the expansion proceeds, and in this case from (1b), (3), and (4) one gets that $\Theta \to \sqrt{3\Lambda}$ if anisotropy is also suppressed, i.e., $\sigma \to 0$. However, Θ is no more bounded from below in the general case, so that in general one can also expect recollapse, *even after an inflationary phase.* From (1b) and (2) we see that if $\Lambda \ge \frac{1}{3}\Theta^2$ then $^{(3)}R > 0$, and also if $\dot{\Theta} > 0$ then $^{(3)}R > 0$, while if $\Lambda \le \frac{1}{3}\Theta^2$ then $\dot{\Theta} < 0$. In particular, this implies that $\Theta < -\sqrt{3\Lambda}$ is a no-return region: every trajectory entering it will undergo collapse with $^{(3)}R/\Theta^2 \to 0^+$. But if a patch of the Universe starting with $\Theta < \sqrt{3\Lambda}$ has to inflate and isotropize, then in order to get the $\Theta \to \sqrt{3\Lambda}$ asymptotic value it has to *superinflate* [20], with $\dot{\Theta} > 0$ *and* $^{(3)}R > 0$.

In terms of the usual density parameter $\Omega = \Omega_M + \Omega_V$, with the matter and vacuum density parameters defined as usual, $\Omega_M = 3\rho/\Theta^2$ and $\Omega_V = 3\Lambda/\Theta^2$, it follows from (2) that $\Omega > 1 \Rightarrow {}^{(3)}R > 0$ and $^{(3)}R < 0 \Rightarrow \Omega < 1$, and from (1b) that $\dot{\Theta} > 0 \Rightarrow \Omega > 1$, but the reverse relations do not hold because of the shear.

Now, let us consider the specific case of the dynamics of the $H_{ab} = 0$ models, described by system (1). For $\Lambda = 0$ this system has a single stationary point [21], given by the origin in phase space, $\rho = \Theta = \sigma_\pm = \varepsilon_\pm = 0$, and corresponding to Minkowski spacetime. For $\Lambda > 0$ this point bifurcates in 10 new points. Of these, three are static ($\Theta = 0$), with only one physically meaningful (the other two have $\rho < 0$), and represent the Einstein universe, with $\rho = 2\Lambda$ and $\sigma_\pm = E_\pm = 0$. Another point represents the spatially flat de Sitter spacetime, with $\Theta = \sqrt{3\Lambda}$ (assuming expansion), and $\rho = \sigma_\pm = E_\pm = 0$. Two other points have $\sigma_- = E_- = 0$, thus are degenerate and represent an oblate and a prolate configuration expanding exponentially with $\Theta = \sqrt{\Lambda/3}$ and $\Theta = \sqrt{\Lambda}$, respectively. Finally, the two other pairs of stationary points are physically equivalent replicas of these latter two. Now, the linearized stability analysis of these stationary points shows that all of them are saddle points, thus unstable, except the one representing de Sitter, which is asymptotically stable if $\Theta > 0$; i.e., de Sitter *is the unique attractor for expansion.* It then follows that those patches of the $H_{ab} = 0$ models that do not recollapse expand toward a de Sitter phase and isotropize, i.e., $\sigma_\pm \to 0$ and $E_\pm \to 0$, with $\Omega \to 1$ and $^{(3)}R \to 0$. Therefore, even in case the portion of the Universe undergoing inflation was initially negligibly small, in a time scale of order $\alpha = \sqrt{3/\Lambda}$ most of the Universe volume will be isotropized by a phase of de Sitter–type inflation: a picture which is in the spirit of chaotic inflation [1]. According to the inflationary scenario, this de Sitter period will be ended by a suitable reheating process, after which a standard FRW phase will occur with Ω close to unity.

Another view of the phase space for the H_{ab} models is achieved using Ω_M and Ω_V, together with the other dimensionless variables [22] $\Sigma_\pm = \sigma_\pm/\Theta$ and $\varepsilon_\pm = E_\pm/\Theta^2$. Obviously these variables diverge for $\Theta = 0$, so that one has to separately consider the $\Theta > 0$ and $\Theta < 0$ cases. Also, it is convenient to introduce a new "time" $\tau = \pm 3\ln\ell$, using the minus (plus) for the $\Theta < 0$ ($\Theta > 0$) case, so that $d\tau/dt > 0$ in both cases.

Denoting by a prime the derivative with respect to τ, by $\Sigma^2 = 3\Sigma_+^2 + \Sigma_-^2$ the magnitude of the dimensionless shear, and by $\Omega_G = \Omega_M - 2\Omega_V$ the effective gravitational mass density parameter, the evolution equations for our variables for $\Theta < 0$ read

$$\Theta' = \frac{\Theta}{6}(2 + 12\Sigma^2 + \Omega_G), \quad (5)$$

$$\Omega_V' = -\frac{\Omega_V}{3}(2 + 12\Sigma^2 + \Omega_G), \quad (6a)$$

$$\Omega_M' = -\frac{\Omega_M}{3}(12\Sigma^2 - 1 + \Omega_G), \quad (6b)$$

$$\Sigma'_+ = \frac{\Sigma_+}{6}(2 - 12\Sigma^2 - 6\Sigma_+ - \Omega_G) + \frac{1}{3}\Sigma_-^2 + \varepsilon_+ , \tag{6c}$$

$$\Sigma'_- = \frac{\Sigma_-}{6}(2 - 12\Sigma^2 + 12\Sigma_+ - \Omega_G) + \varepsilon_- , \tag{6d}$$

$$\varepsilon'_+ = \frac{\varepsilon_+}{3}(1 - 12\Sigma^2 + 9\Sigma_+ - \Omega_G) - \Sigma_-\varepsilon_- + \frac{1}{6}\Sigma_+\Omega_M , \tag{6e}$$

$$\varepsilon'_- = \frac{\varepsilon_-}{3}(1 - 12\Sigma^2 - 9\Sigma_+ - \Omega_G) - 3\Sigma_-\varepsilon_+ + \frac{1}{6}\Sigma_-\Omega_M . \tag{6f}$$

We see that now the Raychaudhuri equation (5) is decoupled from the rest of the system (6); i.e., all other equations do not depend on Θ. The advantage we get in introducing system (6) is that, in addition to the nonstatic ($\Theta \neq 0$) stationary points of system (1), we also have a set of stationary points representing models for which the variables in (1) diverge, while those in (6) obviously have finite constant values. More precisely, system (6) admits many isolated stationary points, three of which are unphysical ($\Omega_M = -3$), with many degenerate physically equivalent triplets, and two physically equivalent sets of points parametrized by Σ_+. The physically significant and distinct points are listed in Table I; points L I, L II, and L III have $\Omega_V \neq 0$ and are the same nonstatic points of system (1) described above. Points D I–D VI are degenerate points with $\Omega_V = 0$ (i.e., for finite Θ they represent $\Lambda = 0$ models), and the set T III represents triaxial configurations parametrized by Σ_+ (see the notes in Table I).

TABLE I. Stationary points of system (6) (only the physically interesting points are listed). For expansion ($\Theta > 0$) the de Sitter point L I is asymptotically stable, and for collapse ($\Theta < 0$) the attractor is given by the points of the Kasner set, i.e., the T III family and its conjugate $\overline{\text{T III}}$. All other points are saddles. Models marked with M are equivalent to Minkowski.

Point	Ω_M	Ω_V	Σ_+	Σ_-	ε_+	ε_-	Model
L I	0	1	0	0	0	0	de Sitter
L II	0	3	$-1/3$	0	$1/3$	0	prolate
L III	0	9	$2/3$	0	0	0	oblate
D I	1	0	0	0	0	0	Flat FRW
D II	0	0	0	0	0	0	Milne (M)
D III	0	0	$1/6$	0	0	0	Szekeres (M)
D IV	0	0	$-1/3$	0	0	0	Kasner (M)
D V	0	0	$1/3$	0	$2/9$	0	Kasner
D VI	0	0	$-1/12$	0	$1/32$	0	Szekeres
T III	0	0	Σ_+	$\Sigma_-(\Sigma_+)$[a]	$\varepsilon_+(\Sigma_+)$[b]	$\varepsilon_-(\Sigma_+)$[c]	Kasner

[a] $\Sigma_- = \frac{1}{\sqrt{3}}\sqrt{1 - 9\Sigma_+^2}$.
[b] $\varepsilon_+ = \frac{1}{3}\Sigma_+(6\Sigma_+ + 1) - \frac{1}{9}$.
[c] $\varepsilon_- = -\frac{\sqrt{3}}{9}(6\Sigma_+ - 1)\sqrt{1 - 9\Sigma_+^2}$.

The linearized stability analysis shows again that for expansion $\Theta > 0$ point L I, locally representing a flat de Sitter universe, is again the unique attractor, while for collapse, $\Theta < 0$, the set T III, together with its conjugate $\overline{\text{T III}}$ (given by the sign exchange $\Sigma_- \to -\Sigma_-$ and $\varepsilon_- \to -\varepsilon_-$ in the expressions in Table I), is attracting. If collapse occurs $\Theta \to -\infty$ and $\Omega_V \to 0$, while $\Sigma_\pm$ and $\varepsilon_\pm$ tend to finite values, and $\Omega_M \to 0$ as well: it can be shown [13] that the sets T III and $\overline{\text{T III}}$ are locally equivalent to the Kasner models, thus the outcome of the stability analysis for system (6) is that collapsing configurations generically tend to a triaxial Kasner-type spindle singularity, with matter and Λ having no effects in the final stage.

To get a clue of what goes on locally in an inhomogeneous universe with an effective cosmological constant we plot in Fig. 1 three significant inflationary cases of several numerical integrations of system (1): one finally recollapsing, and two expanding, one with ${}^{(3)}R < 0$, and the other with ${}^{(3)}R > 0$. The de Sitter phase is asymptotically approached with $\Theta/\sqrt{3\Lambda} \to 1$, ${}^{(3)}R \to 0$, $\Omega \to 1$, and $q \to -1$. We have found that in many cases $\Theta/\sqrt{3\Lambda}$ gets below the flat de Sitter line, then approaches it from below, asymptotically going as (4), with ${}^{(3)}R \to 0^+$. Related to this, we have observed in our numerical results that ${}^{(3)}R$ does not change sign during the evolution, al-

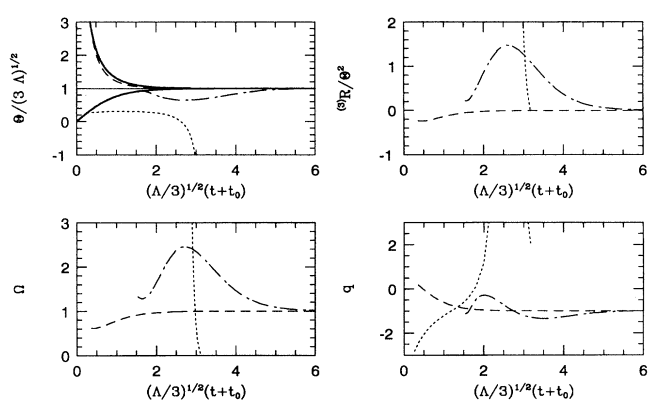

FIG. 1. For three inflationary cases, with one finally recollapsing, we plot four significant dimensionless variables. Top left: $\Theta/\sqrt{3\Lambda}$ against $\sqrt{\Lambda/3}(t + t_0)$, where t_0 and the upper bounds (thick lines) can be read from (3) and (4). The $\Theta = \sqrt{3\Lambda}$ line represents flat de Sitter, the upper bound (3) open de Sitter, and the bound (4) closed de Sitter. In the strip $|\Theta| \leq \sqrt{3\Lambda}$ ${}^{(3)}R > 0$. Top right: the dynamically normalized 3-curvature scalar ${}^{(3)}R/\Theta^2$. Bottom left: $\Omega = \Omega_M + \Omega_V$. Bottom right: the deceleration parameter $q = -3\dot{\Theta}/\Theta^2 - 1$ ($q < 0$ during inflation, $q < -1$ during superinflation, and $q \to 2$ during collapse). Note that for the case with ${}^{(3)}R > 0$, Ω diverges from unity until superinflation occurs, as depicted by q. For the recollapsing case, before diverging at turnaround ($\Theta = 0$), ${}^{(3)}R/\Theta^2$ and Ω are out of scale.

though we cannot formulate this as a rule; also, note that although ${}^{(3)}R \rightarrow \infty$ in the collapse, ${}^{(3)}R/\Theta^2 \rightarrow 0$: the approach to the singularity is Weyl dominated. Finally, we observe that in general Ω does not approach unity even during the inflationary phase, unless superinflation occurs.

In this Letter we analyzed $H_{ab} = \omega_{ab} = p = 0$ models with a cosmological constant $\Lambda > 0$ thought as representing the local value of the vacuum energy density, and possibly causing the occurrence of an inflationary phase in certain patches of the Universe with suitable initial conditions. Of course in a realistic scenario such a cosmological constant will ultimately decay into radiation, reheating the Universe to a standard FRW phase. The picture that emerges is that of a generally inhomogeneous universe, with large patches of it where, thanks to an early inflationary evolution, the local properties are close to that of a flat FRW model. In this picture, however, the observable parameters H_0, Ω_0, and q_0 should be thought of as local values, not to be interpreted as giving the global properties of the Universe. Sure enough, this picture needs to be refined. A first step would be the study of topics connected with the global properties of these spacetimes, e.g., the final fate of the collapsing regions and the formation and nature of the singularities [23], and the emergence of horizons enclosing the quasi-isotropic regions. Ultimately, however, these topics should be studied in the more realistic setting of a scalar field spacetime with no symmetries, in order to investigate if and how the presence of pressure gradients would affect both the collapse and the expansion processes [24].

M. B. would like to thank British PPARC (Grant No. GR/J36440) and Università degli Studi di Trieste for financial support, and SISSA for hospitality during the preparation of this work. S. M. and O. P. acknowledge Italian MURST for financial support.

[1] A. D. Linde, *Inflation and Quantum Cosmology* (Academic Press, New York, 1990); E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Reading, 1990).
[2] M. S. Turner, Report No. FERMILAB-Conf-92/313-A, 1992.
[3] S. W. Hawking and I. G. Moss, Phys. Lett. **110B**, 35 (1982); W. Boucher and G. W. Gibbons, in *The Very Early Universe,* edited by G. W. Gibbons and S. W. Hawking (Cambridge University Press, Cambridge, 1983); J. D. Barrow, *ibid.*
[4] R. M. Wald, Phys. Rev. D **28**, 2118 (1983).
[5] A. A. Starobinski, JETP Lett. **37**, 66 (1983); L. G. Jensen and J. A. Stein-Schabes, Phys. Rev. D **35**, 1146 (1987).
[6] D. S. Goldwirth and T. Piran, Phys. Rep. **214**, 223 (1992).
[7] For an example of this sort of instability in the framework of homogeneous models, see, e.g., V. Sahni and L. A. Kofman, Phys. Lett. A **117**, 275 (1986).
[8] We use units $c = 8\pi G = 1$, and definitions as in [13,19].
[9] A. Barnes and R. R. Rowlingson, Classical Quantum Gravity **6**, 949 (1989).
[10] S. Matarrese, O. Pantano, and D. Saez, Phys. Rev. D **47**, 1311 (1993).
[11] In a recent paper, it is shown explicitly that these constraints are nontrivially satisfied by the evolution equations; W. M. Lesame, P. K. S. Dunsby, and G. F. R. Ellis, Phys. Rev. D (to be published).
[12] S. Matarrese, O. Pantano, and D. Saez, Phys. Rev. Lett. **72**, 320 (1994); Mon. Not. R. Astron. Soc. **271**, 513 (1994).
[13] M. Bruni, S. Matarrese, and O. Pantano, Astrophys. J. (to be published).
[14] G. F. R. Ellis, in *Gravitation,* edited by R. Mann and P. Wesson (World Scientific, Singapore, 1991); P. Coles and G. F. R. Ellis, Nature (London) **370**, 606 (1994).
[15] P. Szekeres, Commun. Math. Phys. **41**, 55 (1975); J. D. Barrow and J. Stein-Schabes, Phys. Lett. **103A**, 315 (1984).
[16] Physically equivalent replicas of these degenerate models are obtained setting $\sigma_- = \pm 3\sigma_+$ and $E_- = \pm 3E_+$.
[17] The expansion scalar Θ, the shear σ_{ab}, and the electric Weyl tidal field E_{ab} are defined with respect to this 4-velocity vector; see [13] and [19] for more details.
[18] Y. Kitada and K. Maeda, Classical Quantum Gravity **10**, 703 (1993).
[19] G. F. R. Ellis, in *General Relativity and Cosmology,* edited by R. K. Sachs (Academic Press, New York, 1971).
[20] F. Lucchin and S. Matarrese, Phys. Lett. **164B**, 282 (1985).
[21] We have used Maple V to find the stationary points and to compute the eigenvalues of the Jacobian.
[22] Similar variables were defined by J. Wainwright and L. Hsu, Classical Quantum Gravity **6**, 1409 (1989).
[23] P. Szekeres, Phys. Rev. D **12**, 2941 (1975).
[24] A. D. Rendall, in *Approaches to Numerical Relativity,* edited by R. D'Inverno (Cambridge University Press, Cambridge, 1992).