PHYSICAL REVIEW LETTERS

VOLUME 74 | 13 MARCH 1995 | NUMBER 11

# Diffusion and Superdiffusion of a Particle in a Random Potential with Finite Correlation Time

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(Received 25 August 1994)

We study theoretically the long time asymptotic of a quantum particle moving in a random time-dependent potential with finite correlation time, in $d = 1$. By applying a new unitary numerical scheme we first show the minor importance of quantum interference and then derive an effective Langevin-type equation for the corresponding clasical problem in the limit of weak potential. We find that on intermediate time scales $\overline{E_{\rm kin}(t)} \sim t^{2/5}$, while the true long time asymptotic is determined by a new friction term, which gives rise to a stationary power law velocity distribution, multifractality of the velocity moments, and a slowing down of the superdiffusive behavior.

PACS numbers: 05.40.+j, 42.25.–p, 71.55.Jv

The motion of a quantum particle in a random time-dependent potential $V(x,t)$ is described by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x,t)\psi\,, \qquad (1)$$

where $V(x,t)$ is assumed to have a translationally invariant correlation function

$$\overline{V(x,t)V(x',t')} = V_0^2 K(x - x', t - t')\,. \qquad (2)$$

Here and below, the bar denotes an ensemble average and the brackets $\langle\cdots\rangle$ the quantum mechanical expectation value. Equation (1) also describes the directed wave propagation in a stationary random scattering medium in the so-called parabolic approximation [1,2]. While this problem was first addressed a long time ago [3,4], it has recently regained much attention [5–10], partially due to a series of new results for the imaginary-time version of the Schrödinger equation (1), i.e., the so-called directed polymer problem [11].

Much numerical study of the Schrödinger equation (1) or closely related lattice models has been performed recently. However, none of these works can be considered as dealing with the true continuum limit of Eq. (1), either due to the absence of strict unitarity [2] or by allowing the velocity to reach the cutoff imposed by the discretization scheme [6,7,10], or due to some additional geometric restrictions to ensure unitarity [9,12]. The unitarity condition of Eq. (1) was realized [6,10] to be a very important constraint which leads to a change of the universality class from that of the directed polymer. It has been shown both numerically [9,10] and analytically [3,9] that for strictly unitary lattice models, the width scales as $\overline{\langle x^2\rangle} \sim t$, and the center position $\overline{\langle x\rangle^2} \sim t^{1/2}$ in $d = 1$ and $\overline{\langle x\rangle^2} \sim \ln^2(t)$ in $d = 2$. Multifractality of the wave function was predicted in [10], although this conclusion was debated in [12]. Additional interesting properties were found when a magnetic field is applied [12].

There also exist several analytical studies of the asymptotic wave packet evolution described by Eq. (1). In an early work [4], it was shown that for $K(x,t) \propto \delta(t)$, a quantum particle experiences rapid stochastic acceleration: $\overline{E_{\rm kin}} \propto \overline{v^2(t)} \sim t$, which leads to $\overline{\langle x^2(t)\rangle} \sim t^3$. In [5], however, it was shown that $\overline{\langle x^2(t)\rangle}$ can grow diffusively if the correlator is a delta function of *both* space and time. Disagreement exists as to what happens when the correlation time is finite. In [7], the scaling law $\overline{v^2(t)} \sim t^{2/5}$ was predicted and it was also argued that such an asymptotic is valid for either a quantum particle or classical one upon invoking the correspondence principle. In [8], in contrast, it was argued that a finite correlation time should not alter the asymptotic $\overline{v^2(t)} \sim t$. Thus there is a general confusion as to what the true asymptotic of Eq. (1) is for finite correlation time. It is also unclear how the numerical results from the various models mentioned above relate to the conflicting analytical predictions.

In this Letter, we attempt to resolve this confusing situation. To this end we develop a new unitary numerical scheme which faithfully represents the continuum Schrödinger equation (1). We also perform extensive simulations for the corresponding classical problem described by Newton's equation, which give results in agreement with the quantum simulation data on large time scales, indicating that quantum interference is unimportant in our problem. Hence analytical predictions based on Newton's equation should apply to the corresponding quantum problem. A new Langevin-type equation is then derived in the limit of weak potential, which shows scaling behavior in agreement with [7], i.e., $\overline{v^2(t)} \sim t^{2/5}$ on intermediate time scales. However, the true long time asymptotic is determined by a new friction term which slows down the above superdiffusive behavior and eventually can give rise to saturation of the average kinetic energy. For simplicity, we focus our attention to the one space dimension ($d = 1$) case.

We first describe our new unitary numerical method to simulate the continuum Schrödinger equation (1). In this method, the wave function $\psi(x,t)$ is propagated by using the following iteration scheme:

$$\psi(x_i, t) = e^{-iV(x_i,t)\tau/\hbar} \sum_j K_F(x_i - x_j)\psi(x_j, t - \tau), \quad (3)$$

where $K_F(x) = (m/2\pi i\hbar\tau)^{1/2}\exp(imx^2/2\hbar\tau)$ is the Feynman free propagator. In the continuum limit $\tau \to 0$ and $x_{i+1} - x_i = a \to 0$, this scheme corresponds to the exact path integral solution of the Schrödinger function. The convolution in Eq. (3) is calculated by using the fast Fourier transform (FFT) method and the Feynman propagator $\hat{K}_F(q) = \exp(-i\hbar q^2\tau/2m)$ in momentum space. The procedure is strictly unitary for arbitrary time step $\tau$, and computationally efficient, since the FFT computation time scales as $N \ln N$, where $N$ is the number of sites in the $x$ direction. Without using the FFT, the scheme (3) has been applied to wave propagation through atmosphere in statistical radiophysics, where it is known as the random phase-screen model [1]. In our simulations, we always make sure that the conditions $a \ll \hbar/mv$ and $\tau \ll \hbar/(mv^2)^{1/2}$ are satisfied, so that the results obtained correspond to the true continuum limit. This is illustrated, for instance, by our ability to reproduce the well-known diffraction pattern $|\psi|^2 \propto \sin^2(x)/x^2$ for $V(x,t) = \text{const}$ and an initially rectangular wave packet. As a further test, we reproduced the previously derived analytical results [4,8] $\overline{\langle v^2 \rangle} \sim t$ and $\overline{\langle x^2 \rangle} \sim t^3$ for an uncorrelated random potential, i.e., $K(x,t) \sim \delta(t)$, as shown in Fig. 1(a). Moreover, in Fig. 1(b) we present the so far unknown result for the scaling of the wave center position $\overline{\langle x \rangle^2} \sim t^3$.

We generate the random potential by superimposing that from individual scattering centers at random ($x_\alpha$, $t_\alpha$), i.e., $V(x,t) = \sum_\alpha V_\alpha g(x - x_\alpha, t - t_\alpha)$, where the amplitudes $V_\alpha$ are also chosen randomly and drawn from a uniform distribution with zero mean. For the individual scattering potential we choose the specific form $g(x,t) = C\exp(-x^2/\xi_x^2 - |t|/\xi_t)$, where $\xi_x$ and $\xi_t$ are the finite correlation length and time and $C$ is a normalization constant. We use a Gaussian wave packet with width $\sigma \gg a$ as the initial wave function. Classical simulations are performed by using the standard finite difference method of Newton's equation for the same random potential as for the corresponding quantum problem.

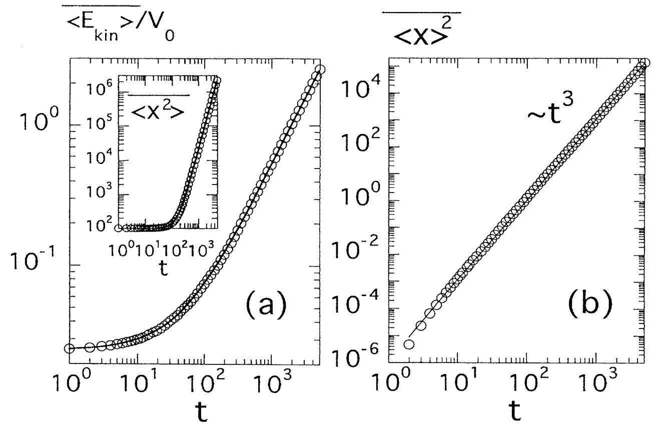


FIG. 1. (a) Kinetic energy $\overline{\langle E_{\text{kin}} \rangle}/V_0$ [inset to (a): the width $\overline{\langle x^2 \rangle}$] and (b) the wave center position $\overline{\langle x \rangle^2}$ as a function of time $t$ for a random potential which is uncorrelated in time, with correlator $K(x,t) = V_0^2[\theta(t) - \theta(t - \tau)]\exp(-x^2/\xi_x^2) \simeq V_0^2\tau\delta(t/\tau)\exp(-x^2/\xi_x^2)$. The parameters used in the simulations are (in units of the lattice constant $a$ and time step $\tau$) $V_0/m = 0.05$, $\xi_x = 10$, $\tau = 1$, $h/m = 6.28$, and $\sigma = 10$. The number of space points is $N = 16\,384$, and averages were taken over 200 realizations of the random potential. The solid lines in (a) correspond to the analytical results found in Refs. [4,8].

The parameters $V_0$, $\xi_x$, and $\xi_t$ give rise to two important velocity scales: $v_0 \equiv \xi_x/\xi_t$ and $v_1 \equiv (2V_0/m)^{1/2}$, such that $v_0 \ll v_1$ represents the strong disorder limit and $v_0 \gg v_1$ that of weak disorder. Figure 2 shows, for both the classical and the quantum problem, the velocity fluctuation as a function of time in (a) the weak disorder limit and (b) that of strong disorder. At large times, $t \gg t_\times$, the classical and quantum data are indistinguishable in the log-log plot and show the scaling behavior predicted in [7], $\overline{\delta v^2}$ (for classical model) $\simeq \overline{\langle \delta v^2 \rangle}$ (for quantum model) $\sim t^{2/5}$ [$\delta v(t) \equiv v(t) - v(0)$]. A simple estimation of the crossover time $t_\times$ yields $t_\times \simeq (v_0/v_1)^4\xi_t$ in case (a) and $t_\times \simeq \xi_t$ in case (b). At short times $t \ll t_\times$, where quantum interference effects may be important, deviations between the classical and quantum data occur in the case of strong disorder [Fig. 2(b)]. The data clearly show that quantum interference is not important at long times, and this justifies the following analytical treatment of Newton's equation.

We now derive an effective Langevin equation for the classical problem. We start with Newton's equation,

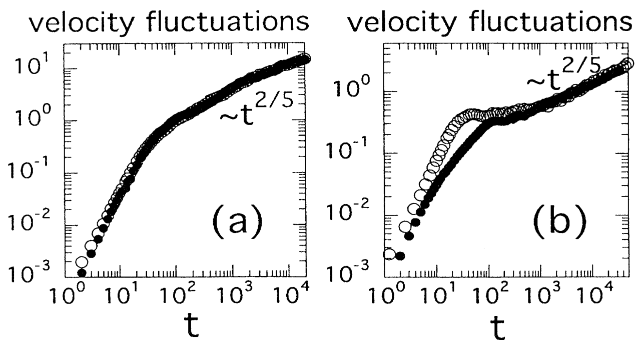


FIG. 2. Velocity fluctuations $\overline{\langle \delta v^2 \rangle}$ (quantum problem, solid circles) and $\overline{\delta v^2}$ (classical problem, open circles) as a function of time for (a) the case of weak disorder, $v_1/v_0 = 1/3$ and (b) that of strong disorder, $v_1/v_0 = 10$ $[\delta v^2(t) \equiv v^2(t) - v^2(0)]$. Parameters used are (a) $V_0/m = 0.056$, $\xi_x = 10$, $\xi_t = 10$, $\sigma = 10$, $h/m = 20$; and (b) $V_0/m = 0.02$, $\xi_x = 4$, $\xi_t = 200$, $\sigma = 10$, and $h/m = 20$. The number of space points in quantum simulations is $N = 16\,384$. Number of realizations in quantum case is 20 and in classical case 600.

$$m\frac{d^2x}{dt^2} = f(x,t), \qquad f(x,t) = -\frac{\partial V(x,t)}{\partial x}, \tag{4}$$

and assume the potential correlator to be of the general form $K(x,t) = K_x(x/\xi_x)K_t(t/\xi_t)$, where $K_x$ and $K_t$ are rapidly decreasing functions. To resolve the question of the true asymptotic scaling behavior, it is enough to consider only a particle with very high velocity, $v(t) \gg v_0, v_1$. In this limit the potential can be considered as weak; i.e., it leads only to a small relative change of velocity $[v(t) - v(t_0)]/v(t)$ during a time interval $(t_0, t_0 + \Delta t)$, where $\Delta t$ is taken to be much larger than $\xi_t$. To zeroth order approximation, the trajectory during this time interval is a straight line $x_0(t) = x(t_0) + v(t_0)(t - t_0)$. This gives $m dv/dt = f[v(t_0)(t - t_0), t]$. In order to take into account the time dependence of the slowly varying velocity on time scales larger than $\xi_t$, we replace $v(0)$ by $v(t)$ (this is correct to leading order of the force amplitude), yielding

$$m\frac{dv}{dt} = f[v(t)(t - t_0), t]. \tag{5}$$

We now show that this equation leads to the stochastic acceleration result $\overline{v^2(t)} \sim t^{2/5}$ [7]. Since $v(t)$ can be regarded as a slowly varying variable on scale $\xi_t$, one can substitute the actual force term on the right hand side of Eq. (5) by a white noise with an intensity equal to the integral of the correlation function of the force:

$$\int dt\, \overline{f(vt,t)f(0,0)} = -V_0^2 \int dt \frac{d^2K(y,t)}{dy^2}\bigg|_{y=vt}. \tag{6}$$

Inserting the general form of the correlator $K(x,t)$ into Eq. (6), we can rewrite Eq. (5) in the following simple form (for $v \gg v_0 = \xi_x/\xi_t$):

$$\frac{dv}{dt} = \frac{A}{v^{3/2}}\,\eta(t), \tag{7}$$

where

$$A^2 = -K_t''(0)\frac{V_0^2\xi_x}{\xi_t^2 m^2}\int dx\, K_x(x), \tag{8}$$

and $\eta(t)$ is a white noise of unit amplitude. Equation (7) coincides with Eq. (7) in [7] for $d = 1$ in the limit $v \gg v_0$. Following [7] we introduce a variable $w(t) = [v(t)]^{5/2}$, and obtain $dw/dt = \frac{5}{2}A\eta(t)$, which leads to $\overline{w^2(t)} \sim t$, and consequently $\overline{v^2(t)} \sim t^{2/5}$. Hence we have found a much simpler way to recover the key result in [7].

What is more important is that Eq. (5) offers a systematic way to make a perturbation expansion with Eq. (7) as the zeroth order approximation. Let $x_1(t)$ be the first order correction to the straight line trajectory, i.e., $x(t) = v(t_0)(t - t_0) + x_1(t)$. Substituting this expression into Eq. (4) gives $m dv/dt = f[v(t_0)(t - t_0) + x_1, t]$. Expanding the force term to first order in $x_1$, and using the zeroth order solution of $x_1(t)$ from Eq. (5), we obtain

$$\frac{dv}{dt} = f[v(t)(t - t_0), t] + \frac{\partial f(y,t)}{\partial y}\bigg|_{v(t)(t-t_0)} \times \int_0^t dt' \int_0^{t'} dt'' f[v(t)(t'' - t_0), t''], \tag{9}$$

where we have replaced $v(t_0)$ by $v(t)$ as before. The terms in this equation are actually the first two terms of an $1/v$ expansion of the initial equation (4), which is rather complicated despite its seemingly simple form. The second term in Eq. (9) is important because—in contrast to the first one—it has a nonzero average, i.e., it corresponds to a *friction* term. Indeed, denoting this second term as $f_1$ we have

$$\overline{f_1} = -\int_0^t dt' \int_0^{t'} dt'' \frac{d^3K(y, t - t'')}{d^3y}\bigg|_{y=v(t-t'')}. \tag{10}$$

The fluctuating part of $f_1$ gives only a $1/v$ correction to the first term $f(vt,t)$ and can be omitted. We also note that due to the replacement of $v(t_0)$ by $v(t)$ in Eq. (9), we have omitted a term of the same order as $f_1$, but this term has zero mean. Using the general form of the correlator, one finds, in the limit $v \gg v_0$ (after several partial integrations), $\overline{f^1} = -B/v^4$ with $B = \frac{3}{2}A^2$. This yields the following Langevin-type equation:

$$\frac{dv}{dt} = \frac{A}{v^{3/2}}\eta(t) - \frac{B}{v^4}. \tag{11}$$

Again using the variable $w = v^{5/2}$, we find

$$\frac{2}{5}\frac{dw}{dt} = A\eta(t) - \frac{B}{w}. \tag{12}$$

It is easy to recognize that Eq. (12) is of the form of a standard Langevin equation $\gamma dw/dt = -dU(w)/dw +$

$\eta_1(t)$, for a Brownian particle moving in a potential $U(w) = U_0 \ln(w)$ with $U_0 = B$, where the noise term obeys $\overline{\eta_1(t)\eta_1(t')} = 2T\gamma\delta(t-t')$ with an effective "temperature" $T = \frac{5}{4}A^2$ [13]. The problem of a particle moving in a logarithmic potential is quite special: The particle cannot escape the potential for any finite temperature $T$, but the equilibrium Gibbs distribution function decays as a power law for large $w$ (as opposed to exponentially): $P(w) \sim \exp[-U(w)/T] \sim w^{-U_0/T}$. This gives the velocity distribution in the original problem:

$$P(v) \sim v^{3/2-5U_0/2T}. \tag{13}$$

We note that $U_0/T = \frac{4}{5}B/A^2 = 6/5$ is universal, i.e., it does not depend on any details of the correlator (2). But this is true only for weak enough potential $v_1 \ll v_0$, because the values of $A$ and $B$ have corrections of the order $(v_1/v_0)^2 \sim V_0$. One can show that to leading order in $V_0$, the correction to $U_0/T$ is positive.

We note that for $U_0/T = 6/5$, the value for $\overline{v^2}$ is infinite, which means physically that the friction term is not strong enough to stop acceleration at large times. But this does not mean that $\overline{v^2} \sim t^{2/5}$ [which corresponds to normal diffusion for $w(t)$] is the true asymptotic scaling law. Indeed, $P(w)$ differs too strongly from that in the free case (a Gaussian) to expect the answer for $\overline{w^2}$ to be still the same for both cases. The new (slower) diffusion law may be obtained from the following speculative argument. For very large times $t$, it is natural to think that the distribution has already reached the Gibbs one, except for some very large distances $w > w_c$. Next, it is reasonable to assume $w_c \sim t^{1/2}$, which corresponds to the free case and can be at least regarded as an upper bound estimate. In this case we have

$$\overline{v^2} = \frac{\int^{w_c} dw\, w^{4/5} \exp[-U(w)/T]}{\int dw \exp[-U(w)/T]} \sim w_c^{4/5+1-U_0/T}$$
$$\sim t^{2/5+(1-U_0/T)/2} \tag{14}$$

for $U_0/T > 1$. (In the opposite case $U_0/T < 1$ we would obtain $\overline{v^2} \sim t^{2/5}$ [7] from this equation instead.) Similarly, the higher moments of $v$ show multifractal scaling,

$$\overline{v^{2n}}^{1/n} \sim t^{2/5+(1-U_0/T)/(2n)}. \tag{15}$$

We have confirmed the key predictions of the theory presented above by alternative analytical means. Equation (11), which plays a central role in our analysis, can be rederived for one particular model of the random potential and by using qualitative arguments. These derivations unfortunately cannot provide exact expressions for $A$ and $B$ and thus will be presented elsewhere. The fact that normal diffusion behavior $\overline{w^2(t)} \sim t$ (corresponding to $\overline{v^2} \sim t^{2/5}$) is not possible for $U_0/T > 1$ can be shown using the Fokker-Planck equation corresponding to Eq. (11), leading to the conclusion that a scaling law slower than simple diffusion for $w$ such as Eq. (14) is expected.

In summary, we have uncovered a set of rich and complicated asymptotic behaviors for the motion of a particle in random time-dependent potential with finite correlation time $\xi_t$ in $d = 1$. First we have found that quantum and classical simulation results agree with each other well at long times. We have shown that the simple scaling $\overline{v^2} \sim t$ [8] is not valid as soon as $\xi_t$ becomes finite. Instead, for up to very long time scales, the asymptotic behavior $\overline{v^2} \sim t^{2/5}$ [7] is expected to hold. The true asymptotic behavior is, however, determined by an effective friction term, which even in the limit of a weak random potential ($v_1 \ll v_0$) is enough to diminish acceleration, leading to deviations from the $\overline{v^2} \sim t^{2/5}$ scaling. This results in the unusual slow diffusion law (14) and multifractality (15). With increasing potential strength $V_0 \sim (v_1/v_0)^2$, the friction term becomes larger, and this probably can stop acceleration at large times, leading to a normal diffusion scaling $\overline{x^2} \sim t$. Such a scenario is consistent with the prediction of normal diffusion for the potential which is delta correlated in both space and time [5,8]. Indeed, the mathematical procedure of taking limits in [5] corresponds physically to the limit $v_1 \to \infty$. It is interesting to mention that a picture rather similar to ours was predicted in the problem of driven Sinai's diffusion [14].

We would like to thank K. Yakubo for valuable discussions, Y. Shapir for sending us the preprint [12], and (P.M.) the Deutsche Forschungsgemeinschaft for financial support. This work was supported in part by the ONR under Grant No. N00014-92-J-4004 (the Medical Free Electron Laser Program) and the NIM Inc.

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