First-Order Phase Transition in Three-Dimensional QED with Chern-Simons Term

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We have studied the chiral phase transition in three-dimensional QED in the presence of a Chern-Simons term for the gauge field. There exists a phase where the chiral symmetry is broken dynamically, and we have determined the critical line for this symmetry breaking as a function of the effective coupling and the strength of the additional Chern-Simons term. In the presence of a Chern-Simons term, the chiral phase transition turns out to be first order, in sharp contrast to the phase transition in pure three-dimensional QED.

Dynamical symmetry breaking in three-dimensional quantum electrodynamics (QED3) has attracted much attention over the last 10 years, both from a purely theoretical field point of view and because of its applications to condensed matter physics in connection with phenomena occurring in planar surfaces [1–5]. A natural extension of pure QED3 is to add a Chern-Simons (CS) term for the gauge field [6], which breaks parity explicitly. Indeed the statistics-changing CS term, together with the question of whether or not there is a dynamically generated fermion mass, plays a key role for variants of QED3 to be effective theories for high-T_c superconductivity and the fractional quantum Hall effect [5]. Furthermore, QED3 also has implications for high energy physics and physics of the early Universe, since three-dimensional models are the high temperature limit of the corresponding four-dimensional theory. Recently, it has been suggested that the effective potential for high temperature QCD is also related to CS gauge theories [7].

The existence of the CS term leads to a novel feature in QED3, namely a *first-order* chiral phase transition, as we show in this Letter. The CS term generates a parity odd mass term for the fermions, but in addition there might be a parity even mass, which breaks chirality [8,9]. Dynamical chiral symmetry breaking, a nonperturbative phenomenon, can be studied using the Dyson-Schwinger (DS) equation for the full fermion propagator. Both a numerical study of the full (truncated) DS equation and an analytical study of the approximated equations show that there is a first-order phase transition. This is quite remarkable and in sharp contrast to the infinite order phase transition one finds using the same truncation scheme in pure QED [1,2].

The Lagrangian in Euclidean space is

$$\mathcal{L} = \bar{\psi}(i\slashed{\partial} + e\slashed{A} - m_e - \tau m_0)\psi + \tfrac{1}{4}F_{\mu\nu}^2$$
$$+ \tfrac{1}{2}i\theta\epsilon_{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \mathcal{L}_{\text{gauge fixing}}, \qquad (1)$$

with the dimensionful parameter θ determining the relative strength of the CS term. We use four-component spinors for the fermions and a four-dimensional representation for the γ matrices. The matrix τ is defined in such a way that the term $m_0\bar{\psi}\tau\psi$ is odd under a parity transformation [9,10]. Also the CS term is odd under a parity transformation, the other terms in the Lagrangian are invariant under a parity transformation.

With such a representation we can define chirality similar to four-dimensional QED. Without an explicit mass m_e for the fermions, the Lagrangian is chirally symmetric, but the parity even mass m_e breaks this symmetry. Note that the other mass m_0 is chirally invariant. Just as in pure QED, chiral symmetry can be broken dynamically due to nonperturbative effects, which can be studied by solving the DS equation for the fermion propagator with both explicit masses m_e and m_0 equal to zero.

The standard way to truncate the DS equation in QED3 is the $1/N$ expansion [11], where N is the number of fermion flavors. The coupling constant e^2 has the dimension of mass, and we use the large N limit in such a way that $e^2 \downarrow 0$ and the product Ne^2 remains fixed: $Ne^2 = 8\alpha$ with α fixed. In this $1/N$ expansion the one-loop vacuum polarization has to be taken into account, because this vacuum polarization is of order 1: Using bare massless fermions, the transverse part of the vacuum polarization is just $\Pi^T(q) = -\alpha|q|$ [1]. It is easy to show that at order 1 there is no parity odd part of the vacuum polarization.

The full vertex is replaced by the bare one, because that is the leading order contribution in $1/N$. In order to be consistent with the requirement that the vertex renormalization and the fermion wave function renormalization are equal, we use a suitable nonlocal gauge function [12]. In pure QED, one can construct a gauge in which the wave function renormalization is exactly 1. In the presence of a CS term, this condition can only be satisfied up to order θ/N, but we are considering small θ only. The proper

choice for the gauge function is [9]

$$a(q) = 2\left(q^2 D^T(q) + \frac{2\alpha}{|q|} + \frac{4\alpha\theta}{q^2} \arctan \frac{\theta|q|}{\alpha^2 + \alpha|q| + \theta^2} \right.$$

$$\left. + \frac{\alpha^2 - \theta^2}{q^2} \ln \frac{\alpha^2 + \theta^2}{(\alpha + |q|)^2 + \theta^2} - 1 \right). \quad (2)$$

With this gauge, we also satisfy the Ward-Takahashi identity up to corrections proportional to θ/N and to the dynamically generated mass function, which are both negligible. Gauge covariance can (in principle) be recovered by applying the Landau-Khalatnikov transformation rules to various Green's functions [13].

The inverse full fermion propagator can be written as

$$S^{-1}(p) = A_e(p)\not{p} + A_0(p)\tau\not{p} - B_e(p) - B_0(p)\tau. \quad (3)$$

The functions $A(p)$ and $B(p)$ are scalar functions of the absolute values of the momenta, and their bare values are $A_e = 1$, $A_0 = 0$, $B_e = m_e$, and $B_0 = m_0$. We use the decomposition $A_\pm = A_e \pm A_0$ and $B_\pm = B_e \pm B_0$, which leads, together with the above truncation scheme, to the following two sets of coupled integral equations:

$$A_\pm(p) = 1 \pm \frac{8\alpha}{Np^2} \int \frac{d^3k}{(2\pi)^3} \frac{2B_\pm(k)D^O(q)}{k^2 A_\pm^2(k) + B_\pm^2(k)} \frac{p \cdot q}{|q|}, \quad (4)$$

$$B_\pm(p) = \frac{8\alpha}{N} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 A_\pm^2(k) + B_\pm^2(k)}$$

$$\times \left[B_\pm(k)\left(2D^T(q) + \frac{a(q)}{q^2}\right) \right.$$

$$\left. \mp 2A_\pm(k)D^O(q)\frac{k \cdot q}{|q|} \right], \quad (5)$$

where D^T and D^O are the transverse and the parity odd part of the gauge boson propagator

$$D^T(q^2) = \frac{q^2 + \alpha|q|}{(q^2 + \alpha|q|)^2 + \theta^2 q^2}, \quad (6)$$

$$D^O(q^2) = \frac{-\theta|q|}{(q^2 + \alpha|q|)^2 + \theta^2 q^2}, \quad (7)$$

and $q = k - p$. Note that the equations for A_+ and B_+ decouple from the ones for A_- and B_-. It is also important to observe that once we have found a solution for A_+ and B_+, we *automatically* have also a solution for A_- and B_-: namely the set $A_- = A_+$ and $B_- = -B_+$. That means that we can always construct a chirally symmetric (but parity odd) solution, with $B_e = 0$. The question of dynamical chiral symmetry breaking turns into the question of whether or not there exist *two* (or more) solutions of the set of integral equations.

Without the CS term there is dynamical chiral symmetry breaking only for $\lambda > \lambda_c = 3/16$ [2], where we have defined the effective coupling $\lambda = 8/N\pi^2$. We expect a similar situation in the presence of the CS term, at least if the parameter θ is small. That means that for $\lambda < \lambda_c$ we only have the chirally symmetric solution of the above

equations, but for $\lambda > \lambda_c$ we expect that there are (at least) two solutions for both B_+ and B_- possible, in such a way that there is a nonzero solution for B_e. An essential difference from pure QED is that in the presence of the CS term there is no trivial solution $B_\pm(p) = 0$. Because of the explicit breaking of parity, the fermions always acquire a parity odd mass term B_0, even if the explicit odd mass term m_0 is zero.

Firstly, we solve the DS equation analytically after some further approximations. Using $A(p) = 1 + O(\theta)$ due to the nonlocal gauge, we replace $A(p)$ by 1, so we get an integral equation for $B_\pm$ only, consisting of two terms, Eq. (5) with $A_\pm(p) = 1$. The first term is the same as in pure QED3, and the essential region for this term is the infrared [1], $p, k \ll \alpha$. So we consider the integral for small momenta only, expand the integration kernel in powers of p and k, and introduce a cutoff at $k = \alpha$. We also linearize the equation by replacing the denominator $k^2 + B_\pm^2(k)$ by $k^2 + B_\pm^2(0)$, which is reliable as long as $B_\pm(p)$ is almost constant for small momenta. In pure QED these approximations lead to almost the same result as the full nonlinear integral equation [14,15].

The second term, proportional to θ, can be calculated by neglecting $B_\pm^2$ with respect to k^2 in the denominator and expanding the integrand in powers of $\min(p,q)/\max(p,q)$ and θ. Taking into account only the leading order terms gives in the infrared region

$$I_\theta(p) = \mp \frac{8\alpha}{N} \int \frac{d^3k}{(2\pi)^3} \frac{2D^O(q)}{k^2 + B_\pm^2(k)} \frac{k \cdot q}{|q|}$$

$$\simeq \pm \lambda\theta + O(p) + O(\theta^3), \quad (8)$$

and in the ultraviolet region $p > \alpha$

$$I_\theta(p) \simeq \pm \frac{11\lambda\alpha\theta}{9p} + O(1/p^2) + O(\theta^3). \quad (9)$$

This means that in the ultraviolet region the CS term will dominate, since without the CS term $B(p)$ falls off much more rapidly in the far ultraviolet. Higher order contributions in $\min(p,q)/\max(p,q)$ will slightly change this result, but not affect the general behavior [15].

Thus we have for $p < \alpha$ and to order θ

$$B_\pm(p) = \frac{4}{3}\lambda \int_0^\alpha dk \frac{k^2}{k^2 + M_\pm^2} \frac{B_\pm(k)}{\max(p,k)} \pm \lambda\theta, \quad (10)$$

where we have defined $M_\pm = B_\pm(0)$. This integral equation can easily be solved by converting it to a second-order differential equation with boundary conditions. The solution is

$$B_\pm(p) = M_\pm \, {}_2F_1(a_+, a_-, \tfrac{3}{2}; -p^2/M_\pm^2), \quad (11)$$

where $a_\pm = \frac{1}{4}(1 \pm i\sqrt{16\lambda/3 - 1})$. The ultraviolet boundary condition leads to the condition

$$M_\pm \, {}_2F_1(a_+, a_-, \tfrac{1}{2}; -\alpha^2/M_\pm^2) = \pm\lambda\theta. \quad (12)$$

In order to determine the behavior of M as a function of θ, we can plot the left-hand side of Eq. (12) divided by λ for a given value of λ as function of the mass parameter M, see Fig. 1 (for convenience we have set $\alpha = 1$ in our figures, which just defines the energy scale). From this figure we can see that there are three solutions possible for B_+ and B_- at small values of θ. A closer look at the region around the origin would reveal that there exist more solutions for extremely small values of θ. In the absence of a CS term there are infinitely many oscillating solutions [1], but it has been shown that the vacuum corresponds to the nodeless one, with the highest value for $|M|$. With the CS term, there is only a finite number of oscillating solutions [9,15].

The chirally symmetric solutions consist of the combination of $B_+(p)$ and $B_-(p)$ with $B_-(p) = -B_+(p)$, which can always be constructed. A solution which breaks chiral symmetry can only be constructed if there is a different solution $\tilde{M}_-$ of Eq. (12), corresponding to $\tilde{B}_-(p)$ which is *not* equal to $-B_+(p)$. As we can see from Fig. 1, this is only possible for $\theta < \theta_c(\lambda)$, beyond this critical value there is only one solution possible for B_+ and B_- which automatically gives $B_e = 0$ and $B_0 = B_+ = -B_-$. At the critical value $\tilde{M}_\pm$ does not become zero, nor does $B_e(0) = [B_+(0) + \tilde{B}_-(0)]/2$, which can be regarded as the order parameter of the chiral phase transition. This clearly signals a first-order phase transition, in sharp contrast to the pure QED case.

We can also plot $M_\pm$ vs λ for a fixed value of θ, see Fig. 2. Here we see that if we increase λ for fixed θ, the chiral symmetry breaking solutions appear if λ exceeds some critical value $\lambda_c(\theta)$, which increases rapidly as a function of θ. This figure shows that the chiral phase transition is first order in this direction as well: Increasing λ beyond $\lambda_c(\theta)$ gives rise to the second (and third) solution, but at the phase transition neither $\tilde{M}_\pm$ nor $B_e(0)$ become zero. In this figure we can also see
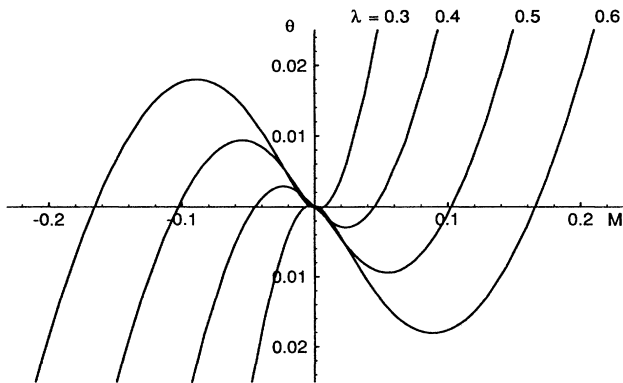
that in the limit $\theta \to 0$ both $M_\pm$ and $\tilde{M}_\pm$ go towards the nontrivial solution m of pure QED, and the critical value λ_c goes towards $\lambda_c(\theta = 0) = 3/16$.

The critical parameters λ_c and θ_c can be calculated directly from Eq. (12). In Fig. 3 we have shown the critical line in the (λ, θ) plane. For small values of M we can make an expansion in order to get an explicit expression for $\theta_c(\lambda)$. To leading order in $\sqrt{16\lambda/3} - 1$, this gives

$$\theta \sim \exp(-3\pi/\sqrt{16\lambda/3 - 1}).   (13)$$

Secondly, we have solved numerically the set of coupled integral equations for A and B, Eqs. (4) and (5), *without any further approximations,* and these numerical results are qualitatively in good agreement with our analytical results. First we checked our assumption that $A_\pm(p)$ is close to 1, and it turns out that the deviation is indeed negligible for small values of θ [15]. Furthermore, we have found numerically the following solutions for B_+, using the notation $m(p)$ for the solution with $\theta = 0$:

1. $B_+(p) = O(m(p)) > 0$ for $\lambda > \lambda_c(\theta = 0)$,

   $B_+(p) = O(\theta)$ for $\lambda < \lambda_c(\theta = 0)$;

2. $\tilde{B}_+(0) = O(-m(0)) < 0$.

The first solution exists for all values of both λ and θ, whereas the second one exists only for values of $\lambda > \lambda_c(\theta)$ and $\theta < \theta_c(\lambda)$. Note that this second solution behaves like θ/p in the ultraviolet, so it has some node at a particular value of p. The existence of this second solution allows for a nonzero chiral symmetry breaking solution $B_e(p) = [B_+(p) + \tilde{B}_-(p)]/2$.

Numerically, it is extremely difficult to establish a first-order phase transition and to determine the critical values λ_c and θ_c. However, our numerical results all support our analytical results and indicate strongly that the chiral phase transition is indeed first order. In Fig. 4 we have shown the behavior of $B_+(0)$ and $\tilde{B}_-(0)$ at fixed θ as a function of λ. We can see that the behavior is the same as in Fig. 2: Increasing λ at fixed θ leads to a second solution

FIG. 1. θ as function of M for several values of λ. The upper half of the plane corresponds to solutions for B_+, the lower half for B_-. Note that both the upper and lower halves correspond to positive values of θ.

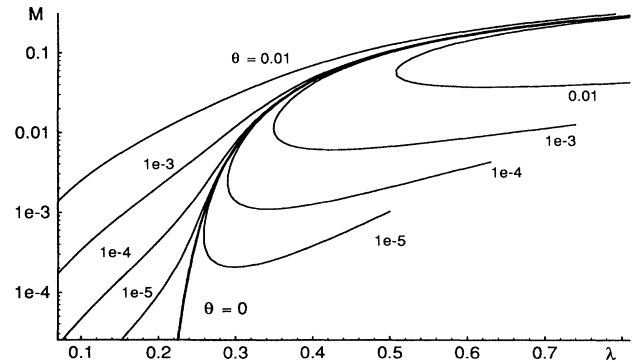
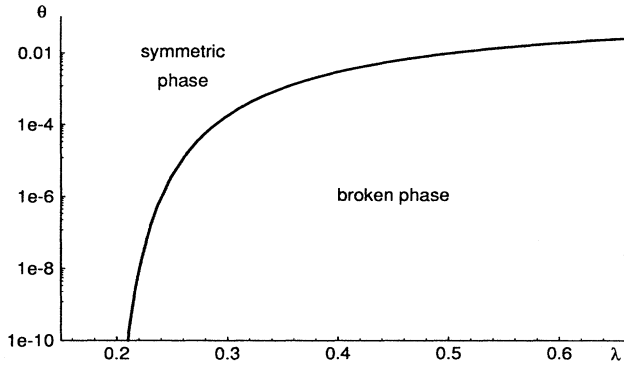FIG. 2. $|M_\pm|$ as function of λ for several values of θ.

FIG. 3.   The critical line for the chiral phase transition in the (λ, θ) plane.

$\tilde{B}_\pm$ beyond some critical value $\lambda_c > \lambda_c(\theta = 0)$. Close to the critical value $\tilde{B}_\pm(0)$ does not go to zero, nor does $B_e(0)$, signaling a first-order chiral phase transition. Also the behavior for increasing θ at fixed λ is qualitatively the same as our analytical result.

In conclusion, both the numerical and analytical results show that there is a first-order chiral phase transition in QED3 with an explicit CS term. This result is very remarkable, given the well-known infinite order phase transition (Miransky scaling) [1] in the absence of the CS term. Also the other known chiral phase transitions in four-dimensional gauge theories are of second (or higher) order. This first-order phase transition is a new and interesting phenomenon, and it might lead to new insights into chiral phase transitions in general. In particular, the connection between the CS term and the first-order phase transition should be studied in more detail.
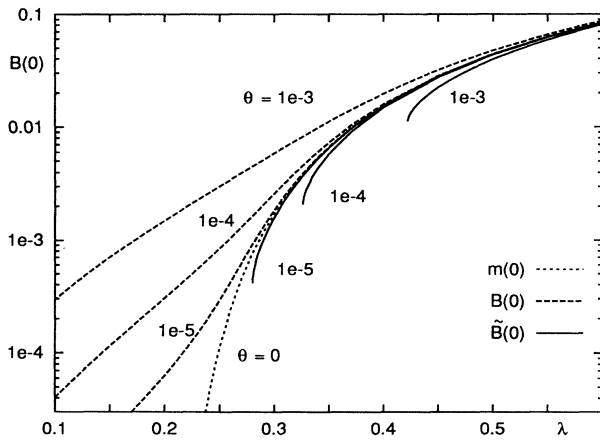

FIG. 4.   The infrared values $B_+(0)$ and $\tilde{B}_-(0)$ as functions of λ for several values of θ.

This result should also be contrasted with some previous results in analyzing this model [8], indicating just a minor quantitative effect on the critical coupling and scaling behavior due to the CS term. Both our numerical and analytical results reveal that the presence of an explicit CS term changes the nature of the chiral phase transition drastically.

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