

## Radiation Reaction in a Continuous Focusing Channel

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We show that the radiation damping rate of the transverse action of a particle in a straight, continuous focusing system is independent of the particle energy, and that no quantum excitation is induced. This absolute damping effect leads to the existence of a transverse ground state to which the particle inevitably decays and yields the minimum beam emittance that one can ever attain,  $\gamma\epsilon_{\min} = \hbar/2mc$ , limited only by the uncertainty principle. Because of adiabatic invariance, the particle can be accelerated along the focusing channel in its ground state without any radiation energy loss.

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In an electron or positron storage ring the amplitude of transverse oscillations damps towards a stable closed trajectory. This damping is caused by the emission of synchrotron radiation due to the uniform bending fields and by the replacement of the energy in the longitudinal direction only. The damping time is approximately equal to the time it takes to radiate away the initial energy of the particle. This damping is counteracted by random fluctuations generated by the discrete photons emitted by each electron, which leads to an equilibrium beam emittance when the damping and excitation rates cancel [1,2].

Radiation damping and excitation are, in principle, present in a straight magnetic or electric focusing system because particles with finite amplitude are bent back towards the straight line trajectory. However, these effects may be modified because the fields are not uniform in such a focusing system. Motivated by these considerations and also by proposals for accelerating charged particles in crystals [3,4], in this paper we study the radiation reaction effect on a charged particle undulating in a straight, continuous focusing system.

Consider an electrostatic focusing channel that provides a transverse continuous potential  $V(x) = Kx^2/2$  for a charged particle, say, a positron, where  $K$  is the focusing strength. The parabolic potential could be, for example, an approximation of the Lindhard potential in the case of planar crystal channeling [5,6]. For simplicity, we take  $x$  as the single transverse dimension of the particle, which has relativistic energy  $E = \gamma m$  and which moves freely (without acceleration) in the longitudinal  $z$  direction with a constant momentum  $p_z = \gamma m \beta_z$  in the absence of radiation. We set  $e = \hbar = c = 1$  in most equations, but reinsert them when suitable. The effect of the additional transverse dimension will be discussed later. We consider the case in which the peak transverse momentum in one oscillation  $p_{x,\max} \ll p_z$ . Defining  $E_z = \sqrt{m^2 + p_z^2}$ , we can approximate the total energy,  $E = \sqrt{m^2 + p_z^2 + p_x^2} + V(x)$ , as  $E_z + E_x$ , where  $E_x = p_x^2/2E_z + V(x)$  is the so-called transverse energy. The motion of the particle is now decoupled into two parts:

a free relativistic longitudinal motion and a transverse harmonic oscillation with an effective mass  $E_z$ .

We now move straight to quantum mechanical analysis of the system because we want to calculate the full radiation reaction including damping and excitation due to discrete photon emissions. Work on relativistic crystal channeling has shown that the spin degree of freedom plays a negligible role [7]. Therefore we use the Klein-Gordon equation to describe the general wave function  $\Psi(x, z, t)$  of the channeled particle,

$$[(-i\nabla - \mathbf{A})^2 + m^2]\Psi = (i\partial_t - V)^2\Psi. \quad (1)$$

In the absence of radiation, we let  $\mathbf{A} = \mathbf{0}$  and look for the energy levels  $E$  and the stationary states  $\Psi(x, z, t) = e^{-iEt}|n, p_z\rangle$  of Eq. (1) by neglecting terms of the order  $(E_x/E)^2$  [8]. We find

$$E \approx E_z + E_x = \sqrt{m^2 + p_z^2} + \omega_z(n + 1/2), \quad (2)$$

$$|n, p_z\rangle = (C_n/L)^{1/2}(E_z\omega_z)^{1/4}e^{ip_z z}e^{-E_z\omega_z x^2/2}H_n(\sqrt{E_z\omega_z}x), \quad (3)$$

where  $C_n = (2^n n! \sqrt{\pi})^{-1}$ ,  $L$  is the length of the channel,  $E_z = \sqrt{m^2 + p_z^2}$  as before,  $\omega_z = \sqrt{K/E_z}$  is the transverse oscillation frequency,  $n$  is the transverse quantum number ( $n = 0, 1, 2, \dots$ ), and  $H_n$  is the  $n$ th-order Hermite polynomial. It is clear that the transverse energy level  $E_x = (n + \frac{1}{2})\omega_z$  and the transverse state function are controlled by both  $n$  and  $p_z$ .

Coupling between the channeled particle and the radiation field, represented by the vector potential  $\mathbf{A}$  in Eq. (1), leads to spontaneous emission of photons. By choosing Coulomb gauge,  $\nabla \cdot \mathbf{A} = 0$ , and ignoring the  $\mathbf{A}^2$  term (double-photon emission), we arrive at

$$[-\nabla^2 + m^2 + i2\mathbf{A} \cdot \nabla]\Psi(x, z, t) = (i\partial_t - V)^2\Psi(x, z, t). \quad (4)$$

Moving to the interaction representation we write  $\Psi(x, z, t) = \exp(-i\mathcal{H}_0 t)\psi(x, z, t)$ . Identifying  $(\mathcal{H}_0 - V)^2 = (-\nabla^2 + m^2)$ , and neglecting  $\dot{\psi}(t)$  in the expansion of  $(i\partial_t - V)^2\Psi(t)$  in Eq. (4), we obtain

$$\dot{\psi}(t) = e^{i\mathcal{H}_0 t}[(\mathcal{H}_0 - V)^{-1}\mathbf{A} \cdot \nabla]e^{-i\mathcal{H}_0 t}\psi(t). \quad (5)$$

Using first-order, time-dependent perturbation theory (Fermi's golden rule), we obtain the transition rate  $W_{fi}$  for the particle from an initial state  $|n, p_z\rangle$  (with energy  $E$ ) to a final state  $|n', p'_z\rangle$  (with energy  $E'$ ):

$$W_{fi} = 2\pi |M_{fi}|^2 \delta(E - E' - \omega_\gamma), \quad (6)$$

where the matrix element  $M_{fi}$  is defined by

$$|M_{fi}|^2 = |\langle n', p'_z; k_\gamma | (\mathcal{H}_0 - V)^{-1} \mathbf{A} \cdot \nabla |n, p_z; 0\rangle|^2. \quad (7)$$

The vector potential  $\mathbf{A}$  acting on the radiation field creates a photon of momentum  $\mathbf{k}_\gamma$  and energy  $\omega_\gamma$  ( $\omega_\gamma = |\mathbf{k}_\gamma|$ ) with two possible polarizations  $\hat{\mathbf{e}}_1$  and  $\hat{\mathbf{e}}_2$  ( $\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = 0$  and  $\hat{\mathbf{e}}_{1,2} \cdot \mathbf{k}_\gamma = 0$ ). The operator  $(\mathcal{H}_0 - V)^{-1}$  can be approximated as  $\mathcal{H}_0^{-1}$  by neglecting terms of the order  $(E_x/E)$ . Therefore

$$|M_{fi}|^2 \approx \frac{2\pi}{E'^2 \omega_\gamma} \sum_{j=1}^2 |\langle n', p'_z | e^{-i\mathbf{k}_\gamma \cdot \mathbf{x}} (\hat{\mathbf{e}}_j \cdot \nabla) |n, p_z\rangle|^2. \quad (8)$$

The integral over  $z$  in the above equation gives rise to  $\delta(p_z - p'_z - k_{\gamma z})$ , which expresses the conservation of longitudinal momentum. Together with the conservation of energy, this places a tight constraint on the radiation reaction of the particle. In order to conserve longitudinal momentum, we have  $p'_z = p_z - \omega_\gamma \cos\theta$ , where  $\theta$  is the photon emission angle relative to the focusing axis. For the photon energy  $\omega_\gamma \ll E$ , the longitudinal energy  $E_z = \sqrt{m^2 + p_z^2}$  must accordingly decrease by an amount  $\Delta E_z \approx (p_z/E_z)\Delta p_z = \omega_\gamma \beta \cos\theta$ . Since the total energy of the particle is reduced by an amount  $\omega_\gamma$ , its transverse energy  $E_x = E - E_z$  must decrease by  $\Delta E_x = \omega_\gamma(1 - \beta \cos\theta) > 0$ . It follows that  $(n + \frac{1}{2})\omega_z - (n' + \frac{1}{2})\omega'_z = \omega_\gamma(1 - \beta \cos\theta) > 0$ . For a small change in  $E_z$ ,  $\omega'_z = \sqrt{K/(E_z - \Delta E_z)} \approx \omega_z(1 + \Delta E_z/2E_z)$ . Substituting  $\omega_\gamma \beta \cos\theta$  for  $\Delta E_z$ , we obtain an equation that relates the change of the transverse quantum number to the photon energy and its emission angle,

$$(n - n')\omega_z = (1 - \beta \cos\theta)\omega_\gamma + (\omega_\gamma \beta \cos\theta)E_x/2E_z > 0, \quad (9)$$

which is always positive definite. We therefore conclude that both the transverse energy and the transverse quantum number always decrease after a photon emission process for all possible photon angles.

Introducing the harmonic number  $\nu = n - n'$  and the pitch angle of the particle  $\theta_p = p_{x,\max}/p_z \approx \sqrt{2E_x/E_z}$ , we find from Eq. (9) a condition for the photon energy

$$\omega_\gamma \approx \frac{\nu \omega_z}{1 - \beta \cos\theta + \theta_p^2/4} \approx \frac{2\gamma^2 \nu \omega_z}{1 + \gamma^2 \theta^2 + \gamma^2 \theta_p^2/2}. \quad (10)$$

Note that  $\gamma\theta_p$  in the above equation plays the same role as the undulator strength parameter in undulator radiation [9].

The exact form of the transition rate  $W_{fi}$  given by the integral over  $x$  in Eq. (8) is more complex than usual because the initial and the final transverse states have different effective masses. This issue is handled

by expanding the final transverse wave function as a superposition of the initial ones. One can then express  $W_{fi}$  in terms of associated Legendre polynomials and Laguerre functions [10,11]. However, in the "undulator" regime where  $\gamma\theta_p \ll 1$ , the effective mass difference can be ignored for  $\omega_\gamma \ll E$ , and Eq. (8) can be evaluated by the dipole approximation [7] where terms beyond the linear order in  $x$  are neglected. Thus the transition rate is nonzero only if  $n' = n - 1$  (the dipole selection rule) and is simply given by

$$W_{fi} = \frac{2\pi^2 n \omega_z}{E_z \omega_\gamma} \left[ \frac{\cos^2 \phi (\cos\theta - \beta)^2}{(1 - \beta \cos\theta)^2} + \sin^2 \phi \right] \times \delta[(1 - \beta \cos\theta)\omega_\gamma - \omega_z]. \quad (11)$$

Therefore in this regime the rate of change of the particle's total energy due to dipole radiation is

$$\frac{dE}{dt} = \sum_f \int \frac{d^3 \mathbf{k}_\gamma}{(2\pi)^3} (E' - E) W_{fi} = -\frac{2}{3} \frac{r_e K}{mc} \gamma^2 n \hbar \omega_z, \quad (12)$$

where  $r_e = e^2/mc^2$  is the classical electron radius. After identifying  $n\hbar\omega_z$  with the rms amplitude of the oscillating particle in the large- $n$  limit ( $n\hbar\omega_z \approx E_x = K(x^2)$ ), we see that  $dE/dt$  in the above expression is identical to the classical radiation power, which is proportional to  $E^2 F_\perp^2$  ( $F_\perp$  being the transverse focusing field strength).

We have shown from Eq. (9) that the transverse quantum level  $n$  always decreases after a random photon emission. This conclusion is valid for all oscillation amplitudes, although we focus on the undulator regime where  $\gamma\theta_p \ll 1$  to illustrate the unique feature of radiation reaction in a focusing channel. With the dipole transition rate given by Eq. (11), we can calculate the rate of change of the transverse quantum level

$$\frac{dn}{dt} = \sum_f \int \frac{d^3 \mathbf{k}_\gamma}{(2\pi)^3} (n' - n) W_{fi} = -\frac{2}{3} \frac{r_e K}{mc} n. \quad (13)$$

We see that  $n$  damps exponentially with an energy-independent damping constant  $\Gamma_c = 2r_e K/3mc$ . Note that in the case of radiation in a bending magnet, there is an additional term of opposite sign independent of the quantum level in question that represents the excitation of transverse oscillations [2]. That term is absent in Eq. (13) and the radiation damping is absolute because no quantum excitation is induced by random photon emissions. Since the action of the transverse oscillation is  $J_n = E_x/\omega_z = (1 + 1/2)\hbar$ , the decrement of the transverse energy level  $n$  leads to the radiation damping of this action given by  $dJ_n/dt = -\Gamma_c(J_n - \hbar/2)$ .

One can use classical radiation reaction to obtain a similar result for the radiation damping of the transverse oscillation amplitude [12]. However, our treatment shows that it is the action that damps exponentially (the change of energy modifies the amplitude damping). It also clearly shows how to extend the results to the case where  $\gamma\theta_p \gtrsim 1$ . More importantly, the quantum mechanical calculation

above automatically takes into account the full radiation reaction and shows the absence of excitation in this system (a surprising result viewed from the standpoint of electron synchrotrons and storage rings). It is difficult if not impossible to model the radiation effect of discrete photon emissions *classically* for  $\gamma\theta_p \ll 1$ , because the time during which a typical photon is emitted is much longer than the oscillation period in the undulator regime [2].

The excitation-free reaction of radiation comes from the fact that the transverse quantum level must *decrease* after each radiation process. In the longitudinal direction the particle recoils against the emitted photon in order to conserve the longitudinal momentum between the two particles. However, in the transverse direction the existence of the focusing force destroys the momentum balance and suppresses the recoil effect. The external focusing environment absorbs the excess transverse momentum during the process of radiation. In this sense, the radiation reaction of a channeled particle in the transverse dimension is similar to that in the Mössbauer effect [13].

Because of the lack of recoil and excitation in the transverse dimension, the particle damps exponentially to its transverse ground state ( $n = 0$ ), and this ground state is stable against further radiation (energy and momentum conservation forbid further radiation). In the ground state the particle reaches the minimum value of the action  $J_0 = \hbar/2$ . Relating this minimum action to a normalized emittance, we find

$$\gamma\epsilon_{\min} \equiv J_0/mc = \lambda_c/2, \quad (14)$$

where  $\lambda_c = \hbar/mc$  is the Compton wavelength. This minimum is also the fundamental emittance limited by the uncertainty principle.

One can estimate the time needed for a particle to damp to its ground state. Suppose the particle enters the focusing channel with a transverse energy  $(n_i + \frac{1}{2})\omega_z$  satisfying the undulator condition, it reaches the ground state in a time  $t_g \sim \ln(n_i)/\Gamma_c$ . To illustrate the range of damping times, let us consider two extreme examples: crystal channels and conventional focusing devices for accelerators. The channeling strength for a typical crystal channel is  $K \sim 10^{11}$  GeV/m<sup>2</sup>, so  $\Gamma_c \sim (10 \text{ nsec})^{-1}$ . When a 100 MeV particle is initially barely captured by the crystal channel, the transverse energy of the particle is of the order of the maximum channeling potential energy 100 eV, and the corresponding quantum number  $n_i$  is about 500. Thus in the absence of any dechanneling effects such as multiple scattering [14], the time it takes to damp to the ground state is  $t_g \sim 60 \text{ nsec}$ . For a conventional linear focusing device, the focusing strength is about  $K \sim 30$  GeV/m<sup>2</sup>, so  $\Gamma_c \sim (30 \text{ sec})^{-1}$ . The damping time to the ground state in this case depends upon the logarithm of the initial state  $n_i$ , but will usually be several  $e$ -folding times.

Another novel characteristic of this radiation reaction is that the relative damping rate of the transverse action

can be much faster than the relative damping rate of longitudinal momentum, i.e., the radiation reaction is asymmetric in these two dimensions. The rate of change of the longitudinal momentum can be obtained from the energy loss equation, Eq. (12), with the approximation  $p_z \approx E_z \approx E$ . We obtain

$$\left| \frac{1}{p_z} \frac{dp_z}{dt} \right| \approx \frac{1}{E} \left| \frac{dE}{dt} \right| \approx \frac{\Gamma_c}{2} \gamma^2 \theta_p^2, \quad (15)$$

which is less than  $\Gamma_c$  for  $\gamma^2 \theta_p^2 < 2$ . In the undulator regime we have the condition  $\gamma\theta_p \ll 1$ , thus

$$\left| \frac{1}{J_n} \frac{dJ_n}{dt} \right| \approx \Gamma_c \gg \left| \frac{1}{p_z} \frac{dp_z}{dt} \right|. \quad (16)$$

One major consequence of the above inequality is that a particle may lose only a negligible amount of total energy when it is damped to the transverse ground state. By replacing  $n = n_i \exp(-\Gamma_c t)$  and  $\omega_z \approx \sqrt{Kc^2/E}$  in Eq. (12) and integrating over time, we find the final energy retained in the ground state  $n_f = 0$  is

$$E_f = E_i/[1 + (\gamma\theta_p)_i^2/4]^2. \quad (17)$$

Note that Eq. (17) is derived in the undulator regime where  $\gamma\theta_p \ll 1$ . Thus particles that enter the focusing channel with the same initial energy but different initial pitch angles will all end up in the transverse ground state with a very small relative longitudinal energy spread of  $(\gamma\theta_p)_i^2/2$ .

We have shown that the radiation reaction in a straight, continuous focusing channel is fundamentally different from that in a bending magnet. In a uniform magnetic field, the radiating particle recoils against the emitted photon by both reducing its orbital quantum number and by shifting the center of its circular orbit [2]. This latter change is allowed due to the translational invariance of the system in the plane perpendicular to the magnetic field, i.e., the system is degenerate with regard to the orbiting centers. The center shift is even necessary in order that the tangent of the particle trajectory be continuous before and after the emission. Therefore the photon emission yields a random recoil of the electron due to variations in both angle and magnitude of the photon's momentum. The resulting random shifts in the orbit center give rise to the random excitations of radial betatron oscillations.

On the other hand, the existence of a focusing axis in a straight, continuous focusing environment removes such a degeneracy and therefore eliminates any quantum excitation to the particle from random photon emissions. In a conventional storage ring, the stored particles are confined by both bending and focusing fields. However, the focusing field is typically so much weaker than the bending field that its radiation effect is negligible. On the average, radiation damping in a conventional storage ring shrinks the momentum vector of the particle proportionally [1,15].

Nevertheless, the above results of straight, focusing channels can be extended to "quasistraight" systems

provided that the focusing field is much stronger than the bending field. The radiation formation length due to bending is of the order  $\rho/\gamma$  [1,2], where  $\rho$  is the bending radius. When this length is much longer than the betatron wavelength, the transverse damping due to the local oscillations is much faster than that caused by the global bending of the trajectory. In this case the radiation reaction is dominated by the focusing field [11].

We note that all the results obtained here are not affected by adiabatic acceleration along the longitudinal direction, since both the action and the stationary states in our system are adiabatic invariants. The condition for adiabatic acceleration is given by

$$\frac{dE_{\text{accel}}}{dt} \ll \omega_z E \approx \sqrt{KE}. \quad (18)$$

Using the previous examples, we get  $\omega_z E \sim 10^5$  GeV/m for a crystal channel and 2 GeV/m for a conventional focusing device when the energy of the particle is only 100 MeV. Obviously, the above inequality is guaranteed by any foreseeable acceleration mechanism. We conclude that the particle, once damped to its transverse ground state in a continuous focusing channel, can be accelerated adiabatically along the channel without any further radiation loss. Therefore the theoretical minimum transverse emittance can be retained at a much higher accelerated particle energy, and the relative longitudinal energy spread can be reduced through acceleration.

We have left out the other transverse degree of freedom of the particle for the sake of simplicity. If the  $y$  direction is free of any force, the particle radiating a photon with a momentum component in the  $y$  direction must recoil by the same magnitude to conserve total momentum in this direction. In general, quantum excitations are present in a force-free dimension. However, if a continuous focusing force also exists in the  $y$  direction, and if both transverse oscillations satisfy the conditions  $\gamma\theta_p^x \ll 1$  and  $\gamma\theta_p^y \ll 1$ , then it is straightforward to extend the discussion above to both transverse dimensions because radiation reaction effects in the  $x$  and  $y$  directions are completely decoupled. Photons are emitted by changing either  $n_x$  or  $n_y$  by one, and all the previous results apply to both dimensions. In the case where the oscillation amplitude is large in the  $x$  or  $y$  direction, there is some coupling between the two transverse degrees of freedom. But if we define the total transverse energy  $E_{\perp} = p_x^2/2E_z + K_1x^2/2 + p_y^2/2E_z + K_2y^2/2$ , from the conservation of both energy and longitudinal momentum, it follows that  $E_{\perp}$  always decreases after a random photon emission. Combining this with the existence of a focusing axis in the continuous focusing system, we conclude that the particle must damp to a mutual transverse ground state ( $n_x = 0$  and  $n_y = 0$ ) that is stable against further radiation.

The basic results obtained here apply to any straight or quasistraight, continuous focusing system. The excitation-free, asymmetric radiation reaction in such systems is the direct consequence of the kinematic requirements and

does not depend on the various approximations used here. There may be interesting applications of this phenomenon in beam handling, cooling, and acceleration. For example, in a sufficiently low-energy, focusing-dominated electron ring, the absolute transverse damping could perhaps be utilized to obtain ultracool beams in transverse phase space with negligible total energy loss. Proposals of miniature linacs powered by lasers [16] would require very strong mesoscopic focusing systems. The results of this Letter provide a radiation damping mechanism to prevent emittance growth. The existence of a transverse ground state for the accelerated particles might also be quite relevant and important. However, when realistic systems are considered, some of the results shown here may be modified. For instance, if other sources of excitation (multiple Coulomb scattering, imperfections, etc.) are present, then the beam may not reach the minimum emittance. When these additional effects are included, the actual equilibrium beam emittance will depend upon the details of the application considered.

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