Characterization of the Transition from Defect to Phase Turbulence

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For the complex Ginzburg-Landau equation on a large periodic interval, we show that the transition from defect to phase turbulence is more accurately described as a smooth crossover rather than as a sharp continuous transition. We obtain this conclusion by using a parallel computer to calculate various order parameters, especially the density of space-time defects, the Lyapunov dimension density, and correlation lengths. Remarkably, the correlation length of the field amplitude fluctuations is, within a constant factor, equal to the length scale defined by the dimension density.

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Recent advances in laboratory technique [1] and in computer simulation [2–4] have opened up the study of boundary-independent spatiotemporal chaos in large homogeneous sustained nonequilibrium systems [5]. Many fundamental questions remain unanswered about such chaotic systems, e.g., what different states can occur, how transport depends on different states, and what kinds of bifurcations separate one state from another. An especially interesting question is whether ideas from statistical mechanics might be applicable to chaotic nonequilibrium systems in the thermodynamic limit of infinite system size [6–8].

A significant step towards understanding some of these questions was recently reported by Shraiman *et al.* [2]. These researchers studied different spatiotemporal chaotic states of the one-dimensional complex Ginzburg-Landau equation

$$\partial_t u(x,t) = u + (1 + ic_1)\partial_x^2 u - (1 - ic_3)|u|^2 u, \qquad (1)$$

on a large periodic interval of length $L = 1024$, which they assumed to be large enough to approximate the thermodynamic limit of an infinite system size. Here the variables t and x denote time and position, respectively, the complex-valued field $u(x,t) = \rho e^{i\phi}$ has magnitude $\rho(x,t)$ and phase $\phi(x,t)$, and the parameters $c_1 > 0$ and $c_3 > 0$ are real valued. Equation (1) is an important model of spatiotemporal chaos because it is simple, experimentally relevant, and universal [8]. Interesting dynamics are predicted and are observed beyond the Newell line $c_1 c_3 = 1$, since all plane wave solutions of Eq. (1) are linearly unstable to the Benjamin-Feir instability for $c_1 c_3 > 1$ [8].

Shraiman *et al.* summarized their simulations in the form of a phase diagram in the c_1-c_3 parameter plane (Fig. 3 of Ref. [2]). Based mainly on calculations of the density of space-time defects n_D [9], this diagram showed continuous and discontinuous transition lines (analogous to second- and first-order phase transitions) separating defect-turbulent from phase-turbulent states [9]. Of special interest to us is the continuous chaos-to-chaos transition line labeled L_1 in their Fig. 3, which occurs for $c_1 \gtrsim 1.8$. It is somewhat mysterious why the density n_D decreases to zero at an L_1 line that is distinct from the Newell line: In the limit of infinite system size and of infinite time, what prevents defects from forming anywhere to the right of the Newell line ($c_3 > 1/c_1$)? The mystery of the L_1 line can be partly appreciated by trying to reason by analogy to equilibrium statistical physics. Assuming that the chaotic fluctuations of Eq. (1) act as a finite-temperature ergodic noise bath and observing that the derivatives in Eq. (1) represent short-ranged interactions between different parts of the field u, we would not expect distinct phases at finite temperature in one-space dimension [10].

Because so little is known about possible critical phenomena of large homogeneous nonequilibrium systems, and because Eq. (1) is such an important model, we have tried to characterize more carefully the dynamics near the L_1 line for the fixed parameter value $c_1 = 3.5$. By calculating various order parameters over length scales as large as $L \leq 10^6$ and over time scales as large as $T \leq 10^7$, we are able to show below that the change from defect to phase turbulence near the L_1 line is more accurately described as a smooth crossover rather than as a sharp continuous transition with power-law scaling of order parameters [2]. We have also studied whether the dimension density δ (Lyapunov fractal dimension per unit volume) is a useful order parameter for characterizing changes in spatiotemporal chaotic states [3,8]. The dimension density defines a dimension correlation length $\xi_\delta = \delta^{-1/d}$ [8] which is the characteristic size of dynamically independent subsystems of spatial dimensionality d [3]. A comparison of ξ_δ with other characteristic length scales as a function of the parameter c_3 gives the remarkable result that ξ_δ is, up to a constant factor, equal to the spatial correlation length of the field magnitude fluctuations ξ_ρ from the Newell line to beyond the L_1 line. An important resource for these calculations was a CM-5 parallel computer [3], which facilitated the study of much larger space-time and parameter regions than were previously conveniently accessible.

Before discussing our results, we note that for $c_1 = 3.5$, for an integration time of $T = 10^5$, and for a periodic interval of length $L = 1024$, Shraiman *et al.* argued the existence of the L_1 line using two key observations [2]:

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(1) that the density n_D vanished as a power law $n_D \propto (c_3 - \bar{c}_3)^\alpha$ with exponent $\alpha \approx 2$ and with $\bar{c}_3 \approx 0.77 > c_3^{\mathrm{Newell}} = 1/c_1 = 0.286$; and (2) that the correlation time τ of phase fluctuations [2] diverged as a power law also at $\bar{c}_3$, as the inverse of the defect density, $\tau \propto 1/n_D$. For $c_3 < \bar{c}_3$, Shraiman *et al.* observed a less-disordered phase-turbulent regime with n_D empirically equal to zero and with slower-than-exponential decay of temporal correlations [2]. If defects do not occur in the thermodynamic limit, a perturbation theory in the small quantity $\epsilon = c_1 c_3 - 1$ yields a simpler description of phase turbulence near the Newell line, $\epsilon \to 0$. In that limit, Eq. (1) reduces to the Kuramoto-Sivashinsky (KS) equation [8,11]

$$\partial_t \phi = -\epsilon\, \partial_x^2 \phi - \frac{1}{2} c_1^2 (1 + c_3^2)\, \partial_x^4 \phi - (c_1 + c_3)(\partial_x \phi)^2, \quad \epsilon = c_1 c_3 - 1, \tag{2}$$

and the amplitude ρ becomes an algebraic function of a spatial derivative of the phase

$$\rho \approx 1 - (c_1/2)\partial_x^2 \phi. \tag{3}$$

Some of our calculations below provide the first quantitative comparisons of phase turbulence as described by Eqs. (2) and (3) with phase turbulence as empirically observed in Eq. (1).

For the parameter value $c_1 = 3.5$, for a system size $L = 4096$, and for an effective integration time of $T = 10^7$ (after allowing transients of duration 10^4 to decay), we find in Fig. 1 that n_D is finite substantially to the left of the L_1 line as calculated in Ref. [2]. Far to the right hand side of the L_1 line, our data in Fig. 1(a) approximately reproduce the previously reported [2] power-law scaling with exponent $\alpha \approx 2$. Closer to the L_1 line, a least-squares fit of the three-parameter expression $a(c_3 - \bar{c}_3')^\alpha$ to the nine leftmost points gives a much larger exponent $\alpha \approx 6.8$, with an onset of phase turbulence ($n_D = 0$) at $\bar{c}_3' = 0.74 < \bar{c}_3 = 0.77$. Assuming equal errors bars on all data points, we find the chi-square value for the fit to be $\chi^2 = 4.6 \times 10^{-12}$. The increase in the exponent with increased space-time resolution suggests that a power-law scaling is inappropriate. As shown in Fig. 1(b), we find a somewhat better fit of the same data with the functional form

$$n_D = a \exp[-b/(c_3 - \bar{c}_3'')^\alpha], \tag{4}$$

which is the expected behavior for thermodynamic Gaussian fluctuations of the phase gradient $\partial_x \phi$ if large values of the latter are the reason for defect nucleation [2]. If we set $\alpha = 1$, a least-squares fit of Eq. (4) to the nine leftmost data points yields the three parameter values $a = 0.66$, $b = 0.98$, and $\bar{c}_3'' = 0.70 < \bar{c}_3' = 0.74$ with $\chi^2 = 8.1 \times 10^{-13}$. If we set $\bar{c}_3'' = c_3^{\mathrm{Newell}} = 0.286$ to test whether Eq. (4) is consistent with the onset of phase turbulence at the Newell line, a least-squares fit (again to the nine leftmost points) gives the parameter values $a = 0.018$, $b = 0.017$, and $\alpha = 8.8$ with a substantially poorer $\chi^2 = 8.2 \times 10^{-11}$. Our data spanning the crossover region evidently lie too far to the right of the Newell line to determine whether the defect density goes to zero before or at this line.

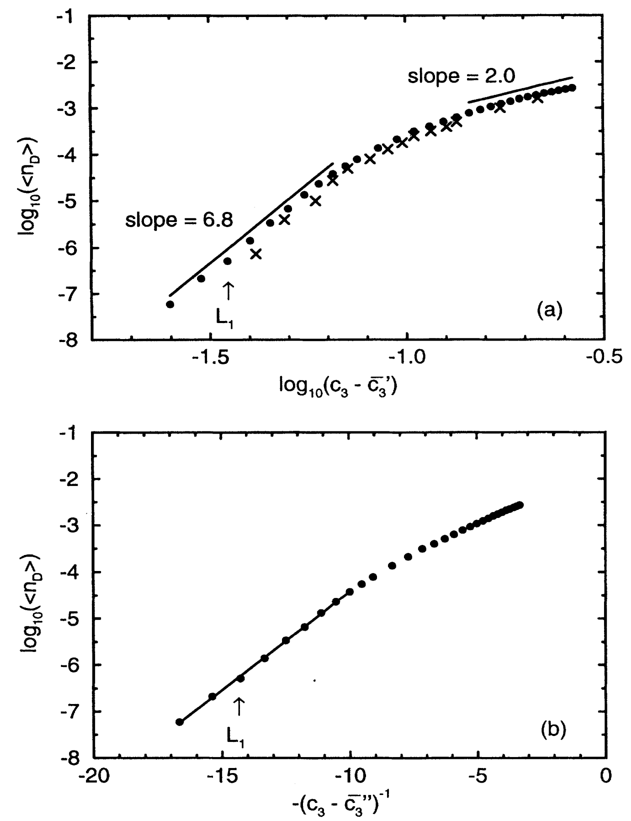


FIG. 1. (a) Log-log plot of the space-time defect density n_D versus the distance $c_3 - \bar{c}_3'$ to the fitted point $\bar{c}_3'$ where the density goes to zero (onset of phase turbulence) for system size $L = 4096$, integration time $T = 10^5$, and an average over 64 randomly specified initial conditions. The numerical time step was $\Delta t = 0.05$. The arrow labeled "L_1" indicates the position of the L_1 line for parameter value $c_1 = 3.5$ [2]. The smallest n_D value corresponds to a count of 200 defects. The two solid lines were drawn to indicate the previous and present best estimates of the exponent α of a power-law scaling. The crosses are the data from Ref. [2]. In (b), we find that Eq. (4) with exponent $\alpha = 1$ gives a better fit of the same data, with an onset of phase turbulence at $\bar{c}_3'' = 0.70$. The straight line is a plot of Eq. (4) over the range of its fit.

To test independently the important implication of Fig. 1 that a crossover occurs, we have calculated other order parameters over the same parameter range. In Fig. 2(a), we show the phase spatial correlation length ξ_ϕ of the quantity $e^{i\phi(x,t)}$ [2,3] as a function of c_3. Shraiman *et al.* argued that ξ_ϕ should be finite in the phase turbulent regime of Eq. (1) and estimated its value indirectly by calculating a phase diffusion coefficient $D = 1/\xi_\phi$ from phase-gradient correlations [2,3]. Exponential decay of spatial correlations is also expected for phase turbulence if the latter is described at long wavelengths by the Kardar-Parisi-Zhang (KPZ) Langevin equation [12]. By going to

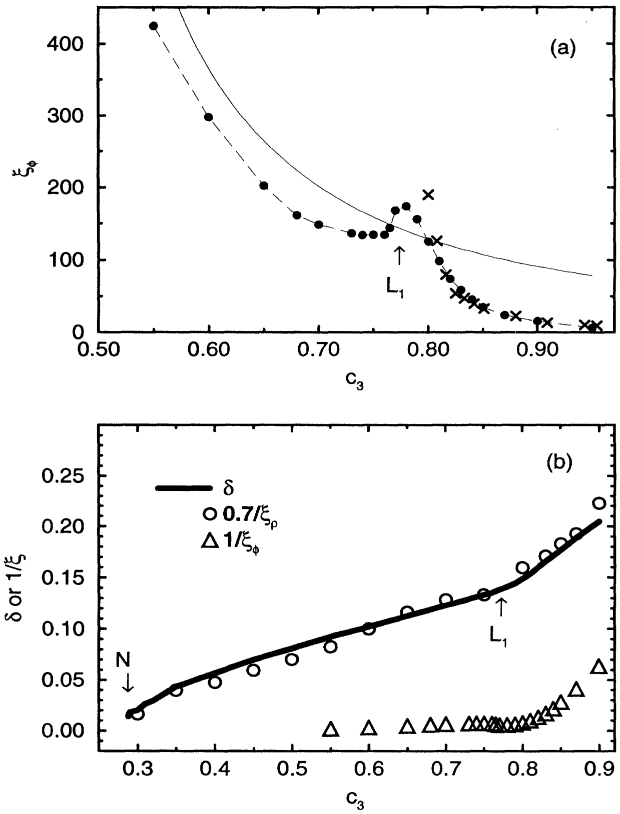


FIG. 2. (a) Plot of the phase correlation length ξ_ϕ for solutions of Eq. (1) for system sizes of up to $L = 10^6$, integration times of up to $T = 2 \times 10^5$, and averages 64 randomly chosen initial conditions. The crosses denote the similar data from Ref. [2]. The solid curve is an analytical expression obtained by scaling the finite correlation length of the parameterless KS equation and is proportional to $\epsilon^{-5/2}$ as $\epsilon \to 0$ [3]. (b) Plot of the Lyapunov dimension density δ [3] and the reciprocals $1/\xi_\rho$ and $1/\xi_\phi$ of the amplitude and phase correlation lengths. The reciprocal length $1/\xi_\rho$ (open circles) has been scaled by a constant factor of 0.7 to emphasize the close agreement with δ. The positions of the Newell and L_1 lines for $c_1 = 3.5$ are denoted by the arrows labeled "N" and "L_1," respectively.

quite large system sizes ($L = 10^6$) and to long integration times, we have verified directly that the phase spatial correlation function decays exponentially well to the left of the L_1 line as shown in Fig. 2(a). As the parameter c_3 decreases, the quantity ξ_ϕ varies *smoothly* through a local maximum near the L_1 line, and then increases steadily until we can no longer estimate its value accurately with our computer resources. The smooth variation of ξ_ϕ through the L_1 region is consistent with a crossover rather than with a sharp transition. The apparent divergence of ξ_ϕ upon approaching the Newell line $\epsilon \to 0$ can be understood semiquantitatively as shown in Fig. 2(a) by a scaling argument [3] that predicts $\xi_\phi \propto \epsilon^{-5/2}$. The agreement is within about 10%.

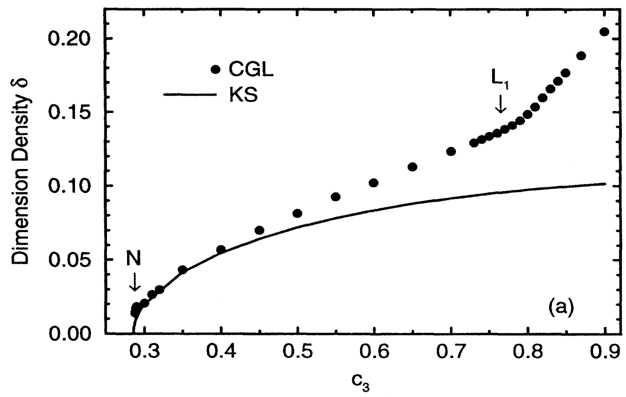
The phase correlation length ξ_ϕ is the same as that of the field u itself [3], but there is a separate, generally shorter, correlation length scale ξ_ρ associated with fluctuations of the field amplitude ρ (also with the phase gradient $\partial_x \phi$). Figure 2(b) compares the reciprocals of the phase and amplitude correlation lengths with the Lyapunov dimension density δ, whose reciprocal defines the dimension correlation length ξ_δ discussed above [3]. Up to a constant factor of 1.4, we find that the amplitude correlation length equals the dimension correlation length ξ_δ over a substantial range of parameter c_3. (An independent and related result was also recently reported by other researchers [13].) *This remarkable result suggests that the big fractal dimension of some large homogeneous chaotic systems might be accurately estimated by simple correlation function calculations.*

In Fig. 3, we make two final comparisons of how phase turbulence, as described by the adiabatic approximation Eq. (3) and by solutions of the KS equation (2), agrees with numerical solutions of Eq. (1). The dimension density δ of the KS equation has been calculated to be $\delta = 0.230$ for the rescaled parameterless version of the KS equation [14] $\partial_t \phi = -\partial_x^2 \phi - \partial_x^4 \phi - \phi \partial_x \phi$. Restoring the original space, time, and magnitude scalings gives the following c_1 and c_3 dependence of the dimension density for KS phase turbulence:

$$\delta = 0.230 \left(\frac{2(c_1 c_3 - 1)}{c_1^2 (1 + c_3^2)} \right)^{1/2} . \tag{5}$$

In Fig. 3(a), we compare Eq. (5) with our empirically determined values of δ for Eq. (1) from Fig. 2(b). The agreement is good up to about $c_3 = 0.5$ ($\epsilon = 0.75$), and then there is an increasing deviation of the actual solutions from Eq. (5). This deviation with increasing c_3 may arise because the adiabatic approximation Eq. (3) breaks down or because higher-order terms in the KS equation are renormalizing the dimension density. Figure 3(b) gives some further insight by comparing the mean-square fluctuation of ρ from Eq. (1) with the mean-square fluctuation of ρ as given by Eq. (3). We observe a previously unreported power-law scaling of these amplitude fluctuations with exponent $\alpha = 4$ from the Newell line to near the L_1 line. Sufficiently close to the Newell line, an exponent of 4 is predicted by rescaling the solutions of Eq. (2). The adiabatic approximation is satisfied over a larger range in c_3 than the agreement between dimension densities.

In conclusion, we have used a parallel computer to characterize more carefully the change from defect to phase turbulence near the L_1 line in the periodic one-dimensional Ginzburg-Landau equation in the limit of large system size. Instead of a sharp continuous transition with power-law scaling of order parameters [2], we found a smooth crossover which suggests that phase turbulence may not exist in the thermodynamic limit of infinite system size. We also found that a short length scale associated with amplitude fluctuations ξ_ρ equals, up to a constant factor, the dimension correlation length ξ_δ associated with the dimension density. This suggests that spatial correlations of certain observables may suffice to estimate



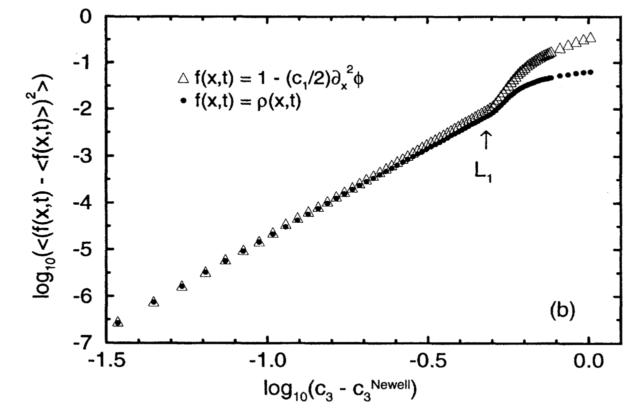


FIG. 3. (a) Comparison of the dimension density δ for solutions for Eq. (1) with the rescaled dimension density of the KS equation, Eq. (5), for $c_1 = 3.5$. (b) Comparison of the mean-square fluctuations of the amplitude ρ as calculated from the one-dimensional CGL equation and as calculated from the adiabatic approximation, Eq. (3), with ϕ determined from Eq. (1). In both (a) and (b), the arrows labeled "N" and "L_1" denote the positions of the Newell and L_1 lines, respectively, for $c_1 = 3.5$.

big fractal dimensions of some large homogeneous chaotic systems. It will be quite interesting to study the generality of these results with further simulations and experiments, especially in two-space dimensions.

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[1] J.P. Gollub and R. Ramshankar, in *New Perspectives in Turbulence,* edited by S. Orszag and L. Sirovich (Springer-Verlag, Berlin, 1990), pp. 165–194; S.W. Morris, E. Bodenschatz, D.S. Cannell, and G. Ahlers, Phys. Rev. Lett. **71**, 2026 (1993); F.T. Arecchi *et al.,* Physica (Amsterdam) **61D**, 25 (1992); Q. Ouyang and H.L. Swinney, Chaos **1**, 411 (1991).

[2] B.I. Shraiman *et al.,* Physica (Amsterdam) **57D**, 241 (1992).

[3] D.A. Egolf and H.S. Greenside, Nature (London) **369**, 129 (1994); D.A. Egolf, Ph.D. thesis, Duke University, 1994.

[4] H.-W. Xi, J.D. Gunton, and J. Viñals, Phys. Rev. Lett. **71**, 2030 (1993); J.E. Pearson, Science **261**, 189 (1993); W. Decker, W. Pesch, and A. Weber, Phys. Rev. Lett. **73**, 648 (1994); H. Chaté, Nonlinearity **7**, 185 (1994).

[5] M.C. Cross and P.C. Hohenberg, Science **263**, 1569 (1994).

[6] P. Gács, J. Comp. Sys. Sci. **32**, 15 (1986); H. Chaté and P. Manneville, Phys. Rev. Lett. **58**, 112 (1987); P.C. Hohenberg and B.I. Shraiman, Physica (Amsterdam) **37D**, 109 (1989); K. Kaneko, Prog. Theor. Phys. Suppl. **99**, 263 (1989); M.S. Bourzutschky and M.C. Cross, Chaos **2**, 173 (1992); J. Miller and D.A. Huse, Phys. Rev. E **48**, 2528 (1993).

[7] L.A. Bunimovich and Y.G. Sinai, Nonlinearity **1**, 491 (1988).

[8] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[9] Defect turbulence is a spatiotemporal chaotic state defined by a finite density n_D of defects (measured in defects per unit time per unit length), where a defect is a space-time point (x, t) at which the field amplitude $\rho(x, t) = |u|$ vanishes and where the phase $\phi(x, t)$ can slip by a multiple of 2π. Phase turbulence is a chaotic state defined by the absence of defects so that $n_D = 0$.

[10] L. Landau and E. Lifshitz, *Statistical Physics* (Pergamon, New York, 1980), 3rd ed., Vol. 5, Part 1, Section 163; J. Fröhlich and T. Spencer, Commun. Math. Phys. **84**, 87 (1982). A correspondence between one-dimensional weakly coupled lattice maps and two-dimensional equilibrium spin systems [7] suggests that distinct nonequilibrium phases are possible in principle. It is not known whether this approach can be generalized and applied to systems like Eq. (1) with continuous space and time variables.

[11] Y. Kuramoto and T. Tsuzuki, Prog. Theor. Phys. **55**, 356 (1976); Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer-Verlag, Berlin, 1984).

[12] G. Grinstein, C. Jayaprakash, and R. Pandit, "Conjectures about Phase Turbulence in the Complex Ginzburg-Landau Equation" (to be published).

[13] T. Bohr, E. Bosch, and W. van de Water, Nature (London) **372**, 48 (1994).

[14] P. Manneville, in *Macroscopic Modeling of Turbulent Flows,* edited by O. Pironneau, Lecture Notes in Physics Vol. 230 (Springer-Verlag, New York, 1985), pp. 319–326.