Phase Effect in Taming Nonautonomous Chaos by Weak Harmonic Perturbations

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An additional weak harmonic forcing is added to the Duffing equation, and its ability of suppressing chaos is analyzed in the global parameter space. It is found that the phase difference between the two sinusoidal forces plays a very important role in suppressing chaos. A new type of intermittency characterized by periodic appearance of regular and chaotic motions, called the breathing effect in this Letter, is observed when the two harmonic forces deviate slightly from resonance.

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Chaos is widely observed in nonlinear systems [1], and this phenomenon is often undesirable in practice. The main technique developed recently of controlling or eliminating chaos is to stabilize one of the unstable periodic orbits embedded in the chaotic attractor, which is referred to as feedback control [2, 3]. Another procedure is to perturb a system parameter by tiny sinusoidal perturbations or to directly add a small external periodic forcing to the chaotic system, which is identified as nonfeedback control [4–11]. Feedback control seems to be efficient but has some difficulties in practice. For instance, it is often difficult in experiments to find the reference unstable periodic states of the unperturbed systems. Moreover, it is also difficult, sometimes, to determine the system state variables which are required for feedback control, because this requires complicated instrumental setups. However, nonfeedback control can be very easily realized in practical systems though the underlying dynamics is still not so clear. Many works have been carried out, by using nonfeedback control, in various chaotic systems analytically [4,5], numerically [4–8], and experimentally [9–11]. In this Letter, we also address the problem of nonfeedback control of chaos by directly adding a weak external forcing to the Duffing equation, as was done by Braiman and Goldhirsch [6], i.e.,

$$\dot{x} = y\,,$$
$$\dot{y} = -\gamma y - x^3 + B\cos(\omega t) + \alpha B\cos(\Omega t + \phi)\,, \tag{1}$$

where we refer to $B\cos(\omega t)$ as the driving force and $\alpha B\cos(\Omega t + \phi)$ as the external control forcing. When the external control forcing is absent, i.e., $\alpha = 0$, the system is in a chaotic state for a certain damping γ and driving B [12]. The object of applying the external control forcing is to lead the system from this chaotic state to a nonchaotic state. To our knowledge, most of the previous studies of taming chaos by nonfeedback control in nonautonomous systems simply set $\phi = 0$ and did not investigate the role played by the phase difference ϕ. Here we fix $\gamma = 0.3$, $\omega = 1$, and $\Omega = 3\omega$, while changing B, α, and ϕ to investigate the control efficiency of the external forcing.

In Fig. 1(a) we show the bifurcation diagram of Eqs. (1) without the external control forcing, i.e., $\alpha = 0$. The chaotic region ranges from $B \approx 7.7$ to $B \approx 12.3$ with a wide period-3 window in-between. All other windows are very small and cannot be clearly seen in our plot. Data in this figure and throughout the presentation are taken on the surface $x = 0$ and $y < 0$ of the section, i.e., the surface of the section is located on the negative y axis. Because there are a number of

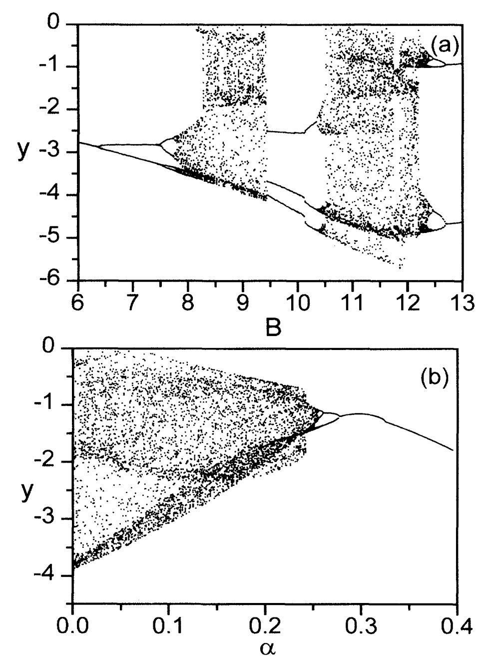

FIG. 1. Bifurcation diagrams of Eqs. (1) with $\gamma = 0.3$, $\omega = 1$, and $\Omega = 3\omega$ (these parameters are fixed and all the data are taken on the surface of the section located on the negative y axis throughout all the plots of this paper). (a) Bifurcation with respect to B without external control forcing ($\alpha = 0$); (b) Bifurcation with respect to α when $B = 8.85$ and $\phi = 0$.

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attractors coexisting for Eqs. (1) in the parameter region investigated here, we integrate Eqs. (1) by employing the conventional technique that the terminal point integrated for the previous parameters is used as the starting point for the sequential parameters, to keep the uniqueness of the simulation result, and all the calculations were started from the same initial point throughout. In Fig. 1(b), we fix $B = 8.85$ and $\phi = 0$, and show the bifurcation with respect to α. Here we only plot the small-α region ($\alpha \leq 0.4$). We find that α is a relevant parameter for bifurcation and chaos. An interesting point is that in the small-α region the external control forcing suppresses chaos preferentially via inverse period-doubling bifurcation. However, in order to reach the inverse period-doubling bifurcation threshold one has to vary the amplitude of the external control forcing to a large extent ($\alpha \geq 0.25$), which is comparable with the changing of B for pushing the system out of the chaotic region via the inverse period-doubling bifurcation. This observation is rather disappointing in the sense of controlling chaos: One has to apply a large external forcing to remove chaos, while by changing B in the same extent without the additional external control forcing, one can also achieve the same purpose.

For controlling chaos one desires that external control forcing with an amplitude very small in comparison with that of the driving forcing should be able to bring the system off the chaotic region. For this purpose, let us examine how the phase difference ϕ of the two forces influences the bifurcation of the system. We still fix $B = 8.85$, while taking $\alpha = 0.075$, and show the bifurcation diagram versus ϕ in Fig. 2(a). It is remarkable that ϕ plays a very important role in suppressing chaos. One observes that a large phase area is dominated by periodic states which connect chaos by period doubling and inverse period doubling to the system. This observation indicates that one can effectively suppress chaos by applying a very weak external control forcing with a properly chosen phase, or in other words, one can use the phase difference of the two forcings to suppress chaos. We refer to this kind of control as phase control of chaos. To have an overview of the control ability of ϕ, we fix $\alpha = 0.06$, and plot the periodic region (period 1, 2, and 4) in the B-ϕ plane [blank in Fig. 2(b)], which leaves the chaotic region (of course, with periodic windows inside) [shaded in Fig. 2(b)] by inverse period-doubling bifurcation. It can be seen that, if one chooses a proper ϕ, chaos can be completely eliminated for the attractor (in the region $7 < B < 13$) with a very small external control forcing. This fact can be seen clearly in Fig. 2(c) where we use $\alpha = 0.075$ and $\phi = 3\pi/2$. One finds that only the period-1 and period-2 states appear, and the whole chaotic region for the attractor we investigate is wiped out for $7 < B < 13$ by this perturbation.

To further demonstrate the effect of the phase difference in suppressing chaos, we plot Fig. 3 [with the same

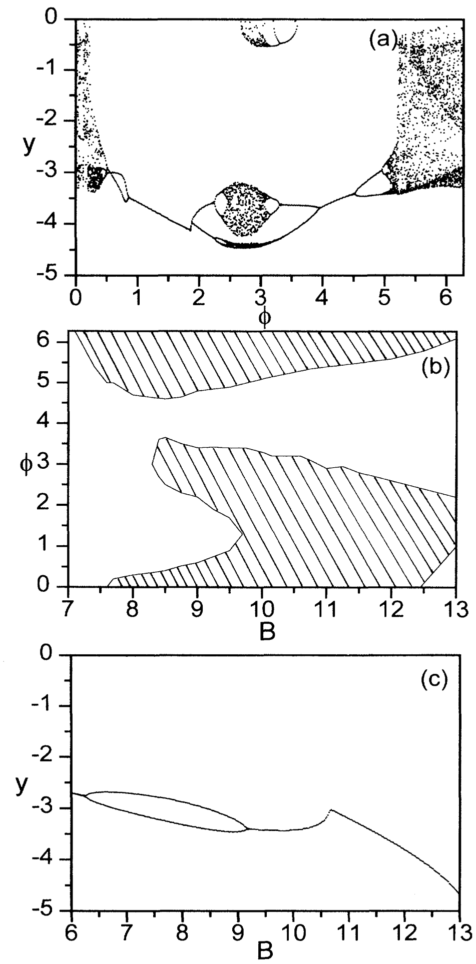

FIG. 2 (a) Bifurcation with respect to ϕ for $B = 8.85$ and $\alpha = 0.075$; (b) Regular motion region (period 1, 2, and 4; blank) and chaotic motion region (including the motions other than period 1, 2, and 4, and including the periodic windows; shaded) in B-ϕ plane with $\alpha = 0.06$; (c) Bifurcation with respect to B as $\alpha = 0.075$ and $\phi = 3\pi/2$.

criterion as in Fig. 2(b)] in the B-α plane as follows. First we fix $\phi = 0$ and evaluate how α influences the bifurcation of the system in the whole B-parameter region. The shaded and the black regions are the chaotic regions including the periodic windows, while the blank regions are regular states which leave the shaded region by inverse period-doubling bifurcation. Then we release the restriction of fixing $\phi = 0$, and change ϕ to obtain the minimum α for which the system leaves chaos by inverse period doubling for a given B. The black region is the region uncontrollable even by changing ϕ. It is remarkable that the black region is considerably contracted from the shaded region, and thus the threshold α of taming chaos can be very much reduced when the phase difference is taken into account.

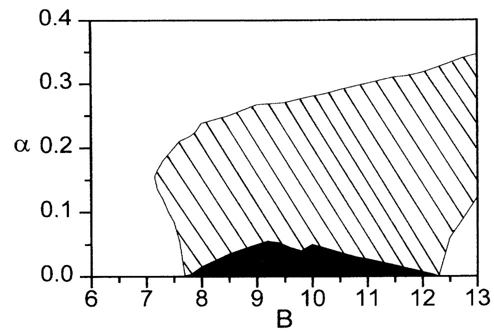

FIG. 3. Blank region corresponds to regular states, while shaded and black regions are chaotic states and periodic windows for the case of $\phi = 0$. The black region is the uncontrollable region even when phase difference ϕ is taken into account. This figure is plotted in the same manner as Fig. 2(b).

Usually, it is very difficult to apply a second frequency just satisfying the exact resonant condition to the first frequency in experiments other than numerical simulations. Very small deviations from resonance may inevitably exist. What will happen if the two frequencies deviate from resonance slightly? To answer this practical question, we assume $\Omega = \Omega_0 + \Delta\Omega$ with $\Omega_0 = 3\omega$ and $\Delta\Omega$ very small. This frequency difference is equivalent to introducing a time-dependent phase difference $\phi(t) = \phi + \Delta\Omega t$ in Eqs. (1). In Fig. 4, we plot the time process of the system at $B = 8.85$, $\alpha = 0.075$, $\Delta\Omega = \frac{1}{3000}$, and $\phi = 0$. Data are obtained on the same surface of the section as above. It is very interesting that in one time interval the system moves *regularly* while in another time segment it moves *chaotically,* and after a time length $T = 2\pi/\Delta\Omega$ the motions are repeated. Therefore, we create a new stable periodic state with period $T = 2\pi/\Delta\Omega$ which includes both regular and chaotic motions in this time interval. By a stable periodic state we mean that the qualitative behavior repeats periodically with period T. However, the trajectory of the system cannot be completely repeated, even on the time scale T, due to the chaotic segments in the evolution process, and thus the motion is not periodic. We identify this kind of motion as a breather. The resemblance of Fig. 4 to Fig. 2(a) is meaningful. Because $\Delta\Omega \ll \Omega$ and the motion is quasistatic, the bifurcation with respect to time in Fig. 4 may well repeat the bifurcation with respect to ϕ in Fig. 2(a). It is remarkable that we find an interesting new state of the system in which chaotic motion and regular motion appear alternately. However, this state, a new type of intermittency, is neither the conventional intermittency (type I, II, and III) nor quasiperiodic motion. On one hand, the alternations of chaotic and *quasiperiodic* segments in the breather appear regularly and periodically, which is essentially different from the standard intermittency. On the other hand, the chaotic segments in the breather are absent in any conventional quasiperiodic state. The dynamics of the new type of intermittency is due to the quasistatic drift in the phase $\phi(t)$; this drift comes from the small detuning. As $\Delta\Omega$ increases the phase drift gradually loses quasistaticity, and the breathing effect gradually becomes ambiguous. In the case of $\Delta\Omega > \frac{1}{100}$ with the other parameters the same as in Fig. 4, one can no longer see this kind of breather. Actually, experimental evidence of this kind of intermittency has been shown by Fronzoni, Giocodo, and Pettini [9] in suppressing chaos in a bistable magnetoelastic beam system. However, the mechanism has not been clearly described. One of us, Qin, and his co-workers have carried out a circuit experiment which demonstrates the behaviors shown in Fig. 4—breathers alternating among *regular* motions and chaotic motions, without any ambiguity [13].

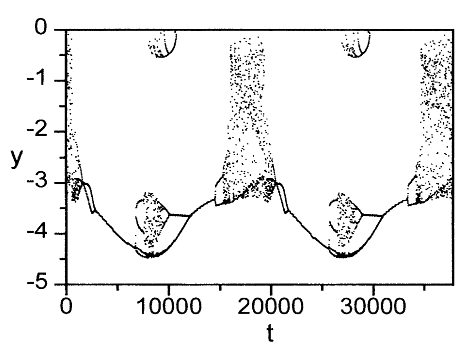

FIG. 4. y versus time for $B = 8.85, \alpha = 0.075$, and $\Delta\Omega = \frac{1}{3000}$.

In the above discussion, we use only $\Omega = 3\omega$ as an example. Actually, those features mentioned above may be kept in the more general case of $\Omega = (q/p)\,\omega$, with p and q being some integers. We have actually tested $\Omega = \frac{1}{2}\omega$, 2ω, 3ω, 4ω, 5ω, 6ω, and got similar results. In addition, we have also carried out other simulations, such as perturbing the x^3 term instead of directly adding the weak control forcing in Eqs. (1), by adding a third sinusoidal forcing to Eqs. (1), and by assuming a stochastic drift in $\phi(t)$, etc. Details of these investigations will be given in a full presentation.

In conclusion, we have investigated the phase effect in suppressing nonautonomous chaos by adding a weak external control forcing. By properly choosing the phase difference of the two sinusoidal forces, one can greatly reduce the amplitude of the control forcing to achieve effective control of chaos. The phase control can be easily realized in real experiments by employing phase-locking techniques. Thus it is expected to have a great application potential in various practical systems. The breather, unlike the conventional ones [14], is a new phenomenon, and its mechanism is presented clearly. In addition, the bifurcation behaviors in systems with two or more frequencies are of great interests [15,16], and how the phase differences influence the bifurcation of the systems requires further investigation.

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