Absolute and Convective Nature of the Eckhaus and Zigzag Instability

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Beyond the primary bifurcation, hydrodynamic pattern-forming systems often show a transition from a structureless base state to a pattern of parallel lines. At the next bifurcation it is commonly observed that these lines become unstable by the Eckhaus or the zigzag mechanism. In open flow systems, one usually has to distinguish between the absolute and the convective thresholds of instability. This paper investigates how an external flow affects these secondary instabilities. It determines when a spatially periodic line pattern becomes absolutely or convectively unstable with respect to Eckhaus or zigzag perturbations. Experiments to test our theoretical predictions are suggested.

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During recent years, a lot of research activity has been concerned with the study of the absolute and the convective nature of instabilities [1]. These two different types of instability appear in so-called open-flow systems. To understand the onset of instability in an open system, one has to distinguish between the temporal growth behavior of spatially localized and extended perturbations, which define the absolute and the convective threshold of stability, respectively. Examples of stability problems in open systems, which have been extensively investigated both experimentally and theoretically, are the Rayleigh-Bénard convection with an imposed horizontal shear flow [2–4] and the Taylor-Couette flow with an axial through-flow [5–7].

Until now, the absolute and convective behaviors of the primary bifurcation have been investigated, while no attention has yet been directed to the secondary bifurcation. We will fill this gap and investigate how an imposed flow affects the onset of the secondary instability. As for the onset of instability of the basic state, we expect that the secondary instability splits into absolute and convective ones.

We consider an open system where the primary instability results in a stationary spatially periodic line pattern, and where the secondary instability is the Eckhaus (E) [8] or the zigzag (ZZ) [9] mechanism. E leads to a spatially periodic compression and dilatation of the line pattern if its wavelength becomes too small or too large, while ZZ is effective only if the wavelength becomes too large. The generality and the broad observability of the E and ZZ instabilities in pattern formation result from universal symmetry properties. Close above the onset of the primary bifurcation, the dynamics of many systems is well represented by so-called amplitude equations, which in turn give a reasonable description (albeit not always quantitative) of the secondary E and ZZ instabilities. Amplitude equations describe the slow spatiotemporal evolution of the most unstable mode. The numerical values of their coefficients are specific for the physical system under consideration, but the form of the equations is universal as they are based on a few underlying symmetries. In what follows we study how an external flow parallel to the pattern-forming plane modifies the E and ZZ boundaries. We restrict ourselves to weak flows, making the analysis analytically treatable. Later we argue that our results concerning E also apply to systems without an external flow, if traveling waves are generated by a Hopf or a drift bifurcation.

We investigate a hydrodynamic system with large lateral extent (x and y directions) and finite thickness (z direction). If a control parameter, say ε, is below a threshold ε_c the system is in its base state, which is homogeneous and isotropic in the (x, y) plane. The profile of any dynamically relevant field $\Phi(x, y, z, t)$ (e.g., velocity, temperature, etc.) is thus of the form $\Phi = \Phi(z)$. As ε is raised above ε_c, the uniform base state exchanges stability with a forward bifurcating stationary line pattern $\exp(i\mathbf{K} \cdot \mathbf{r})$, where $\mathbf{K} = (K_x, K_y)$ and $\mathbf{r} = (x, y)$. The continuous spatial symmetry is broken in the direction parallel to $\mathbf{K}$, but conserved perpendicular to it. In the absence of a lateral flow, the orientation of the pattern is, of course, degenerate due to the isotropy in the pattern-forming plane. In order to investigate the absolute and convective nature of the E and ZZ instabilities, we impose an external shear flow $\mathrm{Re}U(z)\mathbf{e_x}$, which might be driven by a pressure gradient or by moving boundaries. Here $U(z)$ describes the flow profile, while Re measures its strength and thus the anisotropy of the system. We denote patterns with wave vector $\mathbf{K} = (K, 0)$ as transverse, and those with $\mathbf{K} = (0, K)$ as longitudinal. As an example, the reader may imagine the Rayleigh-Bénard system in a large-aspect-ratio container subjected to a horizontal shear flow. Generically the velcoity field $\mathbf{u}(x, y, z, t)$ enters the dynamics of a transported globally conserved quantity Φ via the convective nonlinearity $(\mathbf{u} \cdot \nabla)\Phi$. Therefore the imposed shear flow field leaves the base state profiles

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$\Phi(z)$ unchanged. However, the evolution equations for the convective deviations $\varphi(x, y, z, t)$ are supplemented by *linear* contributions of the form $\mathrm{Re}U(z)\partial_x\varphi$. Thus for small flow rates, the linear dispersion for the disturbance $\varphi \propto \exp(i\mathbf{k}\cdot\mathbf{r} + st)$ of the base state may be expanded in the form

$$s(\mathbf{k}, \varepsilon, \mathrm{Re}) = s_0(k^2, \varepsilon) - ik_x\mathrm{Re}s_1(k^2, \varepsilon) - (k_x\mathrm{Re})^2 s_2(k^2, \varepsilon) + O(\mathrm{Re}^3), \quad (1)$$

where the s_i are real functions and $\mathbf{k} = (k_x, k_y)$. This result is essentially a consequence of Squire's transformation [10], saying that the imposed shear flow enters the problem in form of the produce combination $k_x\mathrm{Re}$. In what follows the investigation of longitudinal and transverse patterns needs a separate treatment.

We start with longitudinal patterns being of the form $B(x, y, t)\exp(iK_c y) + \text{c.c.}$, where K_c is the wave number at onset of the primary instability. The *linear* envelope equation for B may be obtained by introducing $\mathbf{k} = (\delta k_x, K_c + \delta k_y)$ into Eq. (1) and expanding s for small δk_x, δk_y, and ε. By translating $i\delta k_x \to \partial_x$, $i\delta k_y \to \partial_y$, and scaling time, space, and amplitude appropriately, we get the amplitude equation for the longitudinal pattern

$$(\partial_t + v\partial_x)B = \left[\varepsilon + \left(\partial_y - i\frac{\partial_x^2}{2K_c}\right)^2 + \lambda\partial_x^2 - |B|^2\right]B. \quad (2)$$

Here the $(y \to -y)$ reflection symmetry holds, so all coefficients are real, and $v(\lambda)$ is linear (quadratic) in Re as it is related to the second (third) term on the right-hand side (RHS) of Eq. (1). Linear terms of the form $i(\partial_y - i\partial_x^2/2K_c)\partial_x$ and $i(\partial_y - i\partial_x^2/2K_c)\partial_x^2$ have been suppressed, because they do not contribute to the leading order results presented below. Moreover, a cubic nonlinearity has been added to guarantee saturation. The phase winding solution of Eq. (2) $B = B_k\exp(iky)$ represents a stationary longitudinal line pattern with wave vector $\mathbf{K} = (0, K_c + k)$ and amplitude $B_k = \sqrt{\varepsilon - k^2}$. We investigate the stability of this solution by imposing perturbations of the form $\delta B = \exp(\sigma t + iky)[\delta B_1\exp(i\mathbf{q}\cdot\mathbf{r}) + \delta B_2\exp(-i\mathbf{q}^*\cdot\mathbf{r})]$ (the star denotes complex conjugation) which leads to the dispersion relation

$$\sigma(\mathbf{q}, k) = -|B_k|^2 - \frac{1}{2}(U_+ + U_-) \pm \sqrt{|B_k|^4 + \frac{1}{4}(U_+ - U_-)^2}, \quad (3)$$

$$U_\pm = \left(k \pm q_y + \frac{q_x^2}{2K_c}\right)^2 - k^2 + ivq_x + \lambda q_x^2. \quad (4)$$

The E instability is obtained by putting $q_x = 0$. Since all Re-dependent terms drop out in Eqs. (3) and (4), the classical boundary, $\varepsilon_E = 3k^2$, is unaffected by the flow. To study ZZ perturbations we put $q_y = 0$ and find $\sigma(q_x, k) = -U_+$. Spatially *extended* plane wave perturbations (q_x real) define the *convective* onset k_{Zc}. By an expansion in powers of Re we find at lowest order

$$k_{Zc} = -\lambda K_c + \cdots. \quad (5)$$

Here and in the following the dots abbreviate higher order corrections in Re. In comparison to the classical ZZ boundary, $k_Z = 0$, the *convective* onset is delayed by an $O(\mathrm{Re}^2)$ contribution (recall that λ is quadratic in Re). To determine the *absolute* onset one has to investigate the linear growth of spatially *localized* perturbations. To this end, we solve the initial value problem for $\delta B(x, t)$ with Dirac's delta function being the initial local disturbance. This can be done by the Fourier technique where we evaluate the leading behavior of the inverse transform by the method of steepest descent [4]. The long time asymptotic solution is determined by integrating in a small neighborhood of the actual saddle points q_x^s of the dispersion $\sigma(q_x, k)$. Figure 1 sketches the position of the saddle points in the complex q_x plane and shows how the original path of integration (along the real q_x axis) is deformed to the steepest-descent contours $C_1 + C_2$. The absolute threshold for the ZZ instability at $k = k_{Za}$ is defined by requiring that the real part of $\sigma(q_x^s, k)$ changes sign. For small flow rates Re, the coupled conditions $d\sigma(q_x, k)/dq_x = 0$ and $\Re[\sigma(q_x, k)] = 0$ ($\Re$ real part) therefore yield

$$k_{Za} = \frac{1}{2}\frac{2 - \sqrt{7}}{(3 - \sqrt{7})^{1/3}}(v^2K_c^4)^{1/3} \simeq -0.46(v^2K_c^4)^{1/3}. \quad (6)$$

Thus the delay of the *absolute* onset is proportional to $\mathrm{Re}^{2/3}$. Figure 2 summarizes the stability results for longitudinal patterns. As long as the disturbances are spatially

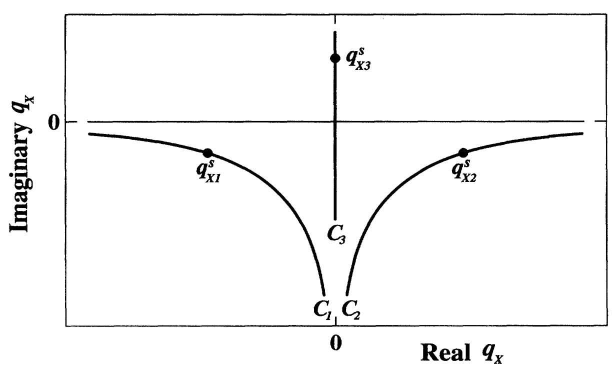

FIG. 1. The saddle points q_{xi}^s, $i = 1, 2, 3$ (defined by $d\sigma/dq_x = 0$) and the corresponding steepest-descent contours C_i (defined by $\Im[\sigma - \sigma(q_x^s)] = 0$) of the dispersion σ as given by Eqs. (3) and (4) with $q_y = 0$. The absolute onset of ZZ perturbations is determined by a Fourier integral of the form $\int(\cdots)\exp(\sigma t + iq_x x)\,dq_x$, evaluated along the real q_x axis. We deform the path of integration to the steepest-descent contours $C_1 + C_2$, so that the asymptotic behavior for ZZ perturbations, as $t \to \infty$, is obtained by integrating in a small neighborhood of the saddled points $q_{x1,2}^s$. The saddle q_{x3}^s does not contribute.

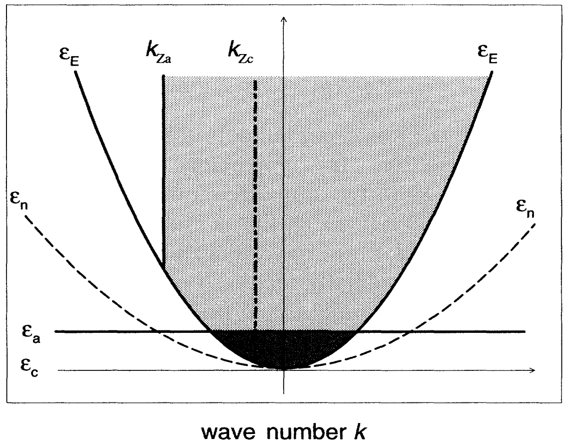

FIG. 2. Stability diagram of the line patterns with wave vector $K_c + k$ aligned parallel to the shear flow. Above the horizontal lines ε_a ($\varepsilon_c = 0$) the homogeneous basic state becomes absolutely (convectively) unstable. $\varepsilon_n = k^2$ denotes the neutral curve and $\varepsilon_E = 3k^2$ the E parabola. The vertical lines $k_{Za} = O(\mathrm{Re}^{2/3})$ and $k_{Zc} = O(\mathrm{Re}^2)$ [see Eqs. (5) and (6)] mark the absolute and convective onsets of the ZZ instability, respectively. In the absence of spatially extended perturbations, line patterns may stably exist in the shaded region. The dark tip of the parabola is a region in which longitudinal line patterns as well as the structureless base state may exist.

localized, stable line patterns may exist in the shaded area. The uniform *base state* is absolutely unstable above the horizontal line $\varepsilon_a = \frac{3}{8}[v^4/(4\lambda)]^{1/3}$ [3,4]. Consequently, there is a small subregion of bistability in the darkly shaded tip of the parabola. In order to test Eq. (6) experimentally, we suggest a Rayleigh-Bénard setup in a long rectangular channel subjected to a horizontal throughflow. If the convective roll pattern is aligned to the flow, the wave number $K_c + k$ can be tuned externally by varying the distance between the sidewalls. By introducing small localized perturbations in the convective bulk region, this technique may be used to detect the onset value k_{Za}.

We now turn to the discussion of the transverse pattern $A(x,y,t)\exp[i(K_c x - \Omega_c t)] + \text{c.c.}$, where $\Omega_c \propto \mathrm{Re}$ is the frequency at the onset of the primary instability. Just as in the derivation of Eq. (2), the amplitude equation for A can be computed by introducing $k = (K_c + \delta k_x, \delta k_y)$ into Eq. (1) and expanding. One finds

$$(\partial_t + v\partial_x)A = \left[(1 + ic_0)(\varepsilon - \varepsilon_c) + (1 + ic_1)\left(\partial_x - \frac{\partial_y^2}{2K_c}\right)^2 + i\chi(1 + ic_3)\partial_y^2 - (1 + ic_2)|A|^2\right]A. \quad (7)$$

The threshold shift ε_c is of second order in Re while all other coefficients are linear. Terms of the form $(1 + ic_4)\partial_y^2\partial_x$ and $i(1 + ic_5)\partial_y^4$ have been dropped, since they do not influence the results below. The phase winding solution $A = A_k \exp[i(kx - \omega t)]$ with $A_k = \sqrt{(\varepsilon - \varepsilon_c) - k^2}$ and $\omega = vk + (\varepsilon - \varepsilon_c)(c_2 - c_0) + (c_1 - c_2)k^2$ represents a traveling transverse line pattern with wave vector $\mathbf{K} = (K_c + k, 0)$ and frequency $\Omega = \Omega_c + \omega$. The stability may be investigated by superimposing perturbations of the form $\delta A = \exp[\sigma t + i(kx - \omega t)][\delta A_1 \exp(i\mathbf{q}\cdot\mathbf{r}) + \delta A_2 \exp(-i\mathbf{q}^*\cdot\mathbf{r})]$, which leads to the dispersion relation

$$\sigma(\mathbf{q},k) = -|a_k|^2 - \frac{1}{2}(U_+ + U_-) \pm \sqrt{|a_k|^4 + \frac{1}{4}(U_+ - U_-)^2 + ic_2|a_k|^2(U_+ - U_-)}, \quad (8)$$

$$U_\pm = \left[\left(k \pm q_x + \frac{q_y^2}{2K_c}\right)^2 - k^2\right](1 \pm ic_1) \pm iv\left(\pm q_x + \frac{q_y^2}{2K_c}\right) \mp i\chi(1 \pm ic_3)q_y^2. \quad (9)$$

For E perturbations ($q_y = 0$) the leading order Re expansion gives (in agreement with [11]) the convective stability boundary (q_x real) at

$$\varepsilon_{Ec} = 3k^2\left[1 + \frac{2}{3}c_2(c_2 - c_1)\right] + \cdots. \quad (10)$$

Depending on the imaginary parts $c_{1,2}$, the correction to $3k^2$ may be either stabilizing or destabilizing. The absolute E boundary (q_x complex) is again determined by the saddle points q_x^s of the dispersion $\sigma(q_x, k)$. The conditions for the onset of instability, $d\sigma/dq_x = 0$ and $\Re[\sigma] = 0$, yield

$$\varepsilon_{Ea} = 3k^2 - \frac{\sqrt{7} - 2}{(3 - \sqrt{7})^{1/3}}\left\{k^4[v + 2(c_1 - c_2)k]^2\right\}^{1/3} \simeq 3k^2 - 0.91\left\{k^4[v + 2(c_1 - c_2)k]^2\right\}^{1/3}. \quad (11)$$

Again, the absolute threshold is delayed by an $O(\mathrm{Re}^{2/3})$ contribution [12]. At this stage it is worth mentioning that the phenomenon of absolute and convective instability is not only restricted to open flows. Amplitude equations of the form of Eq. (7) with $\partial_y = 0$ also occur in closed systems where a Hopf (e.g., in binary-mixture convection) or a drift bifurcation generates traveling waves. Recently the *convective* E boundary (10) has been investigated theoretically and experimentally by Janiaud *et al.* [11]. In the fingering instability [13], one observes a drift instability from a stationary to a traveling pattern. This spontaneous propagation is related to a breakdown of the ($x \to -x$)-parity symmetry of the pattern [14]. The coupled equations for the phase ψ of the pattern and the asymmetry parameter $\mathcal{A}$ lead to a pitchfork bifurcation of $\mathcal{A}$ with a related phase frequency $\partial_t\psi \propto \mathcal{A}$. If one introduces this phase propagation into the evolution equation for the symmetric part S of the order parameter, one arrives again at Eq. (7) with $\partial_y = c_i = 0$ and $v \propto \mathcal{A}$. Provided the coefficients v and c_i are small enough for our Re expansion to be valid, the E stability boundaries

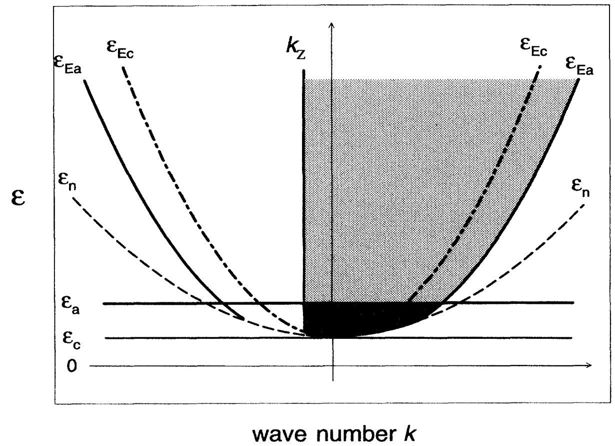

FIG. 3. Stability diagram of traveling line patterns with wave vector $K_c + k$ aligned transverse to the shear flow. Above the horizontal lines ε_a (ε_c) the homogeneous basic state becomes absolutely (convectively) unstable. $\varepsilon_n = k^2$ denotes the neutral curve. The lines ε_{Ec} and ε_{Ea} according to Eqs. (10) and (11) represent the convective and absolute E stability boundaries, respectively. At the vertical line $k_{Zc} = k_{Za} = k_Z$ [Eq. (12)], line patterns become absolutely and convectively unstable to ZZ perturbations. In the absence of spatially *extended* perturbations, line patterns may stably exist in the shaded region. In the dark tip of the parabola, both lines as well as the convetionless state may exist.

given in Eqs. (10) and (11) are expected to apply to those closed systems too. Recently Pan and de Bruyn [13] observed the E instability of nonuniform traveling wave patterns. It will be interesting to see whether the velocity dependent Eckhaus shift predicted by Eq. (11) can be measured in this experiment.

To complete the investigation of transverse patterns we consider now the ZZ instability. By imposing $q_x = 0$ in Eq. (9), one finds a purely real dispersion $\sigma(q_y, k)$. The convective and absolute thresholds therefore coincide, and we obtain

$$k_{Zc} = k_{Za} = k_Z = -\frac{c_2 v}{2} + K_c \chi (c_2 - c_3) + \cdots, \quad (12)$$

i.e., a shift of $O(\mathrm{Re}^2)$. Figure 3 summarizes the stability results for traveling transverse line patterns. Spatially localized perturbations allow stable patterns to exist within the shaded region. Above the horizontal line $\varepsilon_a = \varepsilon_c + v^2/[4(1 + c_1^2)]$ the convectionless base state is absolutely unstable. Thus the dark area is again a region of possible bistability.

We have investigated the absolute and convective onsets of the E and the ZZ instability if a shear flow parallel or perpendicular to the line pattern is imposed. The restriction to weak shear-flow rates (small Re) allows analytical formulas for the threshold shifts. The results concerning logintudinal lines are presented in Eqs. (5) and (6) and Fig. 2, while those for transverse patterns are given in Eqs. (10)–(12) and Fig. 3. The convective onsets are being shifted proportional to Re^2, while the absolute thresholds are delayed by an amount of the order of $\mathrm{Re}^{2/3}$. Since the analysis is performed in terms of amplitude equations, the results are system independent, but confined to the neighborhood of the primary bifurcation.

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