Class of Stationary Axisymmetric Solutions of the Einstein-Maxwell-Dilaton-Axion Field Equations

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A wide class of solutions, endowed with stationary and axial symmetries, of the Einstein-Maxwell-dilaton-axion equations are explicitly given. The chosen coordinate system is such that the structural functions are expressible as a ratio of polynomials of, at most, second degree. It is equipped with six continuous free parameters and two discrete constants. In particular, it contains the generalized Sen black hole with mass, Newman-Unti-Tamburino parameter, charge, angular momentum, dilaton and axion limiting parameters, and related magnetic, dilaton, and axion charges.

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The low-energy effective theory for the heterotic string theory arises as a dimensional reduction and truncations of the string theory in four dimensions under the following considerations (following Sen [1]): compactification of six of the ten dimensions of the string theory and omission of the arising massless fields in the obtained heterotic structure, in this latter, only U(1) charges are permitted; moreover, in the truncated action there are allowed terms containing two or fewer derivatives. Not entering into details and intermediate stages, the dynamical equations of the resulting theory can be deduced from the action [2]

$$S = \int dx^4 \sqrt{-g}\,[R - 2g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}e^{4\phi}g^{\mu\nu}\partial_\mu\kappa\,\partial_\nu\kappa - e^{-2\phi}F_{\mu\nu}F^{\mu\nu} - \kappa F_{\mu\nu}\check{F}^{\mu\nu}], \qquad (1)$$

where R is the scalar Riemann curvature, $g_{\mu\nu}$ is the metric four-dimensional tensor, $F_{\mu\nu}$ is the electromagnetic antisymmetric tensor field, $\check{F}_{\mu\nu}$ its dual ($\check{F}_{\mu\nu} = -\frac{1}{2}\sqrt{-g}\,\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$), ϕ is the dilaton scalar field, and κ is the axion field dual to the three-index antisymmetric tensor field $H = -\exp(4\phi) * d\kappa/4$. The solution of the string sigma model is related to the one of the classical Einstein theory through the metric relation $G_{\mu\nu}(s) = e^{2\phi}g_{\mu\nu}(E)$, where s and E stand, respectively, for string and Einstein.

Because of the complexity of the Einstein-Maxwell-dilaton-axion (EMDA) field equations resulting from the effective action (1), their integration is not a trivial task. Most of the relevant known solutions have been derived by transformations [3–5].

The main objective of this Letter is to give the explicit expression of a wide class of stationary axisymmetric solutions. It contains eleven parameters restricted to three algebraic conditions; thus only eight of them can be considered as free parameters. Moreover, two of the remaining eight free parameters can be scaled to assume independently the discrete values -1, 0, 1. Thus, in general, the obtained EMDA solutions are endowed with six continuous and two discrete parameters. It contains, as a particular case, the generalized Sen solution which is equipped with mass, angular momentum, electric and magnetic charges, dilaton and axion asymptotic constants, and Newman-Unti-Tamburino (NUT) parameter.

This class of solutions has been derived by a straightforward integration process of the EMDA equations for a metric, a canonical one, endowed with stationary and axisymmetric Killing vectors, which has been successfully used previously to derive all aligned electrovacuum type D fields [6], and their generalizations in the presence of a perfect fluid [7]. The fundamental structural functions are rational functions expressible as the ratio of polynomials of, at most, second degree in the coordinate variables. Details of the followed integration procedure will be published elsewhere.

The metric, endowed with the Killing directions ∂_τ and ∂_σ, can be given as

$$ds^2 = \frac{\Delta}{X}dx^2 + \frac{X}{\Delta}(d\tau + N\,d\sigma)^2 + \frac{\Delta}{Y}dy^2 - \frac{Y}{\Delta}(d\tau + M\,d\sigma)^2,$$

$$X = -\epsilon x^2 + 2px + \alpha\,, \qquad Y = \epsilon y^2 + 2\mu y + \alpha\,,$$

$$\epsilon = -1, 0, 1\,,$$

$$M = \nu x^2 + 2bx\,, \qquad \nu = -1, 0, 1\,,$$

$$N = -\nu y^2 + 2\beta y, \qquad \Delta = M - N\,, \qquad (2)$$

where p, μ, b, and β are constants constrained to certain conditions given below. The electromagnetic field $F_{\nu\mu} = A_{\mu,\nu} - A_{\nu,\mu}$ is determined by the electromagnetic 4-vector potential

$$A_\mu = \delta^\tau_\mu A_\tau + A_\sigma\delta^\sigma_\mu\,, \qquad \mu = x, y, \tau, \sigma\,,$$

$$A_\tau = (qy - gx)/\Delta\,, \qquad q = \text{const}\,,$$

$$A_\sigma = \nu yx(qx + gy)/\Delta\,, \qquad g = \text{const}\,. \qquad (3)$$

The dilaton scalar field ϕ is given by

$$\exp(2\phi) = \frac{W}{\Delta} \equiv \omega\frac{(x^2 + y^2)}{\Delta}\,, \qquad \omega = \text{const} > 0\,, \qquad (4)$$

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while the axion field potential occurs to be

$$\kappa = \kappa_0 + 2\frac{by + \beta x}{W}, \qquad \kappa_0 = \text{const}. \tag{5}$$

The parameters appearing in this EMDA class of solutions ought to fulfill the following conditions C,

$$C: \; g\beta - qb = 0, \quad p\beta - \mu b = 0,$$
$$\nu^2 q^2 = 2\omega\beta(\mu\nu + \beta\epsilon), \quad \nu^2 g^2 = 2\omega b(p\nu + \epsilon b), \tag{6}$$

which are written in a symmetric form for further convenience.

Notice that for simultaneously vanishing parameters b and β, $b = 0 = \beta$, the metric and the electromagnetic field reduce to the Carter [A] type D solution, see Ref. [6]; in this case the dilaton and axion fields become constants, and they can be brought to the values $\phi = 0$ and $\kappa = 0$ by assigning to ω and κ_0 the values $\omega = 1$ and $\kappa_0 = 0$. Thus, for EMDA solutions β or either b ought to be different from zero.

The main branches of EMDA solutions, described by the formulas (2)–(6), will be denoted by

$$\begin{aligned} &S(b, \beta \neq 0, \epsilon, \nu, \mu, \alpha, \omega, \kappa_0; p = \mu b/\beta, g = qb/\beta, \\ &\quad \nu q = \sqrt{2\omega\beta(\mu\nu + \beta\epsilon)}) \end{aligned} \tag{7}$$

and

$$\begin{aligned} &S(b \neq 0, \beta, \epsilon, \nu, p, \alpha, \omega, \kappa_0; \mu = p\beta/b, q = g\beta/b, \\ &\quad \nu g = \sqrt{2\omega b(p\nu + b\epsilon)}). \end{aligned} \tag{8}$$

Of course, these branches coincide when both parameters b and β are simultaneously different from zero; they describe different families of solutions when in the first branch $b = 0$, and in the second one $\beta = 0$.

All these EMDA solutions are algebraically general Petrov-type gravitational field. With respect to the null tetrad [in Kramer-Stephani-MacCallum-Herlt (KSMH) formulation [8]]

$$\begin{aligned} &\left.\begin{matrix} m \\ m^* \end{matrix}\right\} = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{\Delta}{X}}\, dx \mp i\sqrt{\frac{X}{\Delta}}\,(d\tau + N\, d\sigma)\right\}, \\ &\left.\begin{matrix} -k \\ l \end{matrix}\right\} = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{\Delta}{Y}}\, dy \pm \sqrt{\frac{Y}{\Delta}}\,(d\tau + M\, d\sigma)\right\}. \end{aligned} \tag{9}$$

The Weyl coefficients occur to be

$$\begin{aligned} \psi_0 &= \psi_4 = 0, \\ 2\Delta^3\psi_1 &= -2\Delta^3\psi_3 = i\sqrt{XY}\,(b^2 + \beta^2), \\ 6\Delta^3\psi_2 &= 6\nu(p\nu + \epsilon b)[(x^2 - 3y^2)x + i(y^2 - 3x^2)y] \\ &\quad + 6\nu(\mu\nu + \beta\epsilon)[(y^2 - 3x^2)y - i(x^2 - 3y^2)x] \\ &\quad + 2[2\epsilon(b^2 + \beta^2) + 3\nu(bp + \beta\mu)](x^2 - y^2) \\ &\quad - 12i[(\epsilon b + p\nu)b + (\nu\mu + \epsilon\beta)\beta]xy \\ &\quad + 4(\beta\mu + bp)(xb + \beta y) + 4\alpha(b^2 + \beta^2). \end{aligned} \tag{10}$$

These quantities are given in a symmetric manner. Depending on the case, here one has to replace $p = \mu b/\beta$ or $\mu = p\beta/b$.

It is clear that for β or either b different from the Weyl coefficients, ψ_1 and ψ_3 do not vanish except for XY becoming zero. In general, the complex curvature coefficient ψ_2 is different from zero for the studied EMDA field, thus the invariants $C(2) = 6(\psi_2)^2 - 8\psi_1\psi_3$ and $C(3)/6 = -(\psi_2)^3$ are different from zero, and consequently the corresponding gravitational field is algebraically general.

Notice that essential singularities arise for the set of points, in which the above quoted invariants tend to infinity, i.e., when Δ vanishes, for

$$\nu(x^2 + y^2) + 2(bx - \beta y) = 0. \tag{11}$$

A particularly relevant EMDA solution is given by the metric (2) subjected to the coordinate transformations

$$\begin{aligned} x &= a\cos\theta - mb/\beta, \qquad y = r, \\ \tau &= t - a\varphi, \qquad \sigma = \varphi/a, \end{aligned} \tag{12}$$

and the parameters chosen as follows:

$$\begin{aligned} &S(b, \beta \neq 0, \epsilon = 1, \nu = 1, \mu = -m, \\ &\quad \alpha = a^2 - m^2b^2/\beta^2, \omega, \kappa_0; \\ &\quad p = -mb/\beta, g = qb/\beta, q^2 = 2\omega\beta(\beta - m)). \end{aligned} \tag{13}$$

In this way we arrive at the EMDA solution given by the metric

$$\begin{aligned} ds^2 = &-\frac{\Sigma - a^2\sin^2\theta}{\Delta}\, dt^2 + \Delta d\theta^2 + \frac{\Delta}{\Sigma}\, dr^2 \\ &+ \frac{d\varphi^2}{a^2\Delta}\left\{a^2\sin^2\theta[r^2 - 2\beta r + a^2]^2 - \Sigma\left[\left(a\cos\theta + \frac{b}{\beta}(\beta - m)\right)^2 - a^2 - b^2\right]^2\right\} \\ &- 2\frac{dt\, d\varphi}{a\Delta}\left\{\Sigma\left[\left(a\cos\theta + \frac{b}{\beta}(\beta - m)\right)^2 - a^2 - b^2\right] + a^2\sin^2\theta[r^2 - 2\beta r + a^2]\right\}, \\ \Sigma = &\; r^2 - 2mr + a^2 - m^2b^2/\beta^2, \\ \Delta = &\; r^2 - 2\beta r - b^2 + \left[a\cos\theta + \frac{b}{\beta}(\beta - m)\right]^2, \end{aligned} \tag{14}$$

the electromagnetic 4-vector potential A_μ,

$$A_x = 0 = A_y\,, \qquad A_t = \frac{1}{\Delta}[qr - g(a\cos\theta - mb/\beta)]\,,$$

$$A_\varphi = \frac{1}{\Delta a}\{-qr[a^2\sin^2\theta + 2mab\cos\theta/\beta - m^2b^2/\beta^2] + g(r^2 + a^2)(a\cos\theta - mb/\beta)\}\,, \tag{15}$$

the dilaton field

$$\exp(2\phi) = \frac{W}{\Delta} \equiv \frac{\omega}{\Delta}\left[r^2 + a^2\cos^2\theta - \frac{mb}{\beta}(2a\cos\theta - mb/\beta)\right]\,, \tag{16}$$

and the axion scalar field κ,

$$\kappa = \kappa_0 + 2\frac{br + \beta(a\cos\theta - mb/\beta)}{W}\,. \tag{17}$$

The asymptotic values of the metric components at infinity ($r \to \infty$) for $b = 0$ (see Ref. [9]) allow one to determine the black hole mass M and the angular momentum J,

$$-g_{tt} \to 1 - 2\frac{M}{r} + O\left(\frac{1}{r^2}\right)\,,$$

$$g_{t\varphi} \to -2\frac{J}{r}\sin^2\theta + O\left(\frac{1}{r^2}\right)\,. \tag{18}$$

From the limiting values of the 4-vector components of the electromagnetic field one determines the electric Q and magnetic P charges, and the magnetic dipole moment μ

$$A_t \to \frac{Q}{r} + \frac{P}{r^2}a\cos\theta + O\left(\frac{1}{r^2}\right)\,,$$

$$A_\varphi \to -\mu\frac{\sin^2\theta}{r} + O\left(\frac{1}{r^2}\right)\,. \tag{19}$$

The asymptotic values of the dilaton and axion fields give rise to their values at infinity ϕ_0 and κ_0 correspondingly, and to the dilaton D_0 and axion A_0 charges according to the expansions

$$e^{2\phi} \to e^{2\phi_0}\left[1 + 2\frac{D_0}{r} + O\left(\frac{1}{r^2}\right)\right]\,,$$

$$\kappa \to \kappa_0 + \frac{2}{r}A_0e^{-2\phi_0} + O\left(\frac{1}{r^2}\right)\,. \tag{20}$$

Thus, in the limit $r \to \infty$, for the studied metric structure (14)–(17) with $b = 0$, one establishes that

$$M = m - \beta\,, \qquad J = a(m - \beta)\,,$$

$$Q = q = \sqrt{2\omega\beta(\beta = m)}\,, \qquad P = g\,, \qquad \mu = qa\,,$$

$$\omega = \exp(2\phi_0)\,, \qquad \kappa_0 = \kappa_0\,, \qquad D_0 = \beta\,. \tag{21}$$

By setting in these expressions $\beta = -m\sinh^2(\alpha/2)$ and $\omega = 1$ one arrives at the parameters characterizing the Sen solution

$$2M = m(1 + \cosh\alpha)\,, \qquad 2J = ma(1 + \cosh\alpha)\,,$$

$$\sqrt{2}\,Q = m\sinh\alpha\,, \qquad \sqrt{2}\,\mu = ma\sinh\alpha\,; \tag{22}$$

see (16) and (17) of Ref. [1].

It is apparent from (14) that the parameter b is related to the NUT parameter $N = p + b = -b(m - \beta)/\beta$, where $p = -mb/\beta$ from (13). Extending the applicability of the asymptotic values at $r \to \infty$ of the metric components and fields to the metric structure (14)–(17) with nonvanishing b, $b \neq 0$, one arrives at

$$M = m - \beta\,, \qquad J = aM\,, \qquad Qe^{\phi_0} := q\,,$$

$$Pe^{\phi_0} := g\,, \qquad \mu = qa\,, \qquad 2D_0 = 2\beta = -Q^2/M\,,$$

$$2A_0 = 2b = P^2/N\,, \qquad \omega = \exp(2\phi_0)\,,$$

$$\kappa_0 = \kappa_0\,, \qquad PM + NQ = 0\,. \tag{23}$$

Thus, the metric structure can be considered as equipped with six free parameters

$$M,\ a,\ Q,\ P\ (\text{or}\ N),\ \phi_0,\ \text{and}\ \kappa_0\,; \tag{24}$$

here we have taken into account the constraint $PM + NQ = 0$ on P and N for independent M and Q parameters. For b different from zero the metric structure (14) shares the same troubles exhibited by the NUT gravitational field in the presence, if any, of an electromagnetic field, i.e., this solution cannot be interpreted properly as a black hole.

One may consider the metric structure (14) as a generalized Sen solution for the EMDA field equations in the string gravity. This structure contains many of the previously known solutions, among them, the Taub-NUT solution [10,11] as a limiting transition of (14) for $a \to 0$.

Results concerned with the most general canonical metric structure (2)–(6), in which, in particular, there are solutions for vanishing discrete parameters ϵ or ν, will be published elsewhere. Moreover, a work concerned with the behavior of the solutions near the black hole is in progress.

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[1] A. Sen, Phys. Rev. Lett. **69**, 1006 (1992).
[2] A. Shapere, S. Trivedi, and F. Wilczek, Mod. Phys. Lett. A **6**, 2677 (1991).
[3] D. Garfinkle, G. Horowitz, and A. Strominger, Phys. Rev. D **43**, 3140 (1991).
[4] G. Gibbons and K. Maeda, Nucl. Phys. **B298**, 741 (1988).
[5] C. Callan, J. Harvey, and A. Strominger, Nucl. Phys. **B359**, 611 (1991).

[6] A. García, J. Math. Phys. (N.Y.) **25**, 1951 (1984).
[7] A. García, J. Math. Phys. (N.Y.) **32**, 708 (1991).
[8] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of the Einstein's Field Equations* (Deutscher Verlag der Wissenschaften, Berlin, 1980).
[9] J.H. Horne and G.T. Horowitz, Phys. Rev. D **46**, 1340 (1992).
[10] C.V. Johnson and R.C. Myers, Phys. Rev. D **50**, 6512 (1994).
[11] R. Kallosh, D. Kastor, T. Ortin, and T. Torma, Phys. Rev. D **50**, 6374 (1994).