

Separation of Variables and Hamiltonian Formulation for the Ernst Equation

D. Korotkin and H. Nicolai

II Institute for Theoretical Physics, Hamburg University, Luruper Chaussee 149, Hamburg 22761, Germany
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It is shown that the vacuum Einstein equations for an arbitrary stationary axisymmetric space-time can be completely separated by reformulating the Ernst equation and its associated linear system in terms of a nonautonomous Schlesinger-type dynamical system. The conformal factor of the metric coincides (up to some explicitly computable factor) with the τ function of the Ernst equation in the presence of finitely many regular singularities. We also present a canonical formulation of these results, which is based on a “two-time” Hamiltonian approach, and which opens new avenues for the quantization of such systems.

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In this Letter we demonstrate that the vacuum Einstein equations for space-time with two commuting Killing vectors can be reformulated in terms of a pair of compatible ordinary matrix differential equations. Similar results can be shown to hold for the more general equations obtained by dimensional reduction from higher-dimensional theories of gravity and supergravity with matter couplings to two dimensions. As a by-product, we establish a previously unknown relation between the conformal factor of the associated metric and the so-called τ function, which plays a pivotal role in the modern formulation of integrable systems [1,2]. Thirdly, we present a canonical formulation of these results, which avoids certain technical difficulties encountered in previous treatments. Our results suggest that an *exact* quantization of axisymmetric stationary (matter-coupled) gravity by exploiting techniques developed for flat space (quantum) integrable systems [3,4] is now within reach.

The Ernst equation and related linear system.—We start from the following metric on stationary axisymmetric space-time [5]:

$$ds^2 = f^{-1}[e^{2k}(dx^2 + d\rho^2) + \rho^2 d\phi^2] - f(dt + F d\phi)^2, \quad (1)$$

where (x, ρ) are Weyl canonical coordinates; t and ϕ are the time and angular coordinates, respectively. The functions $f(x, \rho)$, $F(x, \rho)$, and $k(x, \rho)$ are related to the complex Ernst potential $\mathcal{E}(x, \rho)$ by

$$f = \text{Re } \mathcal{E}, \quad F_\xi = 2\rho \frac{(\mathcal{E} - \bar{\mathcal{E}})_\xi}{(\mathcal{E} + \bar{\mathcal{E}})^2},$$

$$k_\xi = 2i\rho \frac{\mathcal{E}_\xi \bar{\mathcal{E}}_\xi}{(\mathcal{E} + \bar{\mathcal{E}})^2}, \quad (2)$$

where $\xi = x + i\rho$, $\bar{\xi} = x - i\rho$; hereafter subscripts $\xi, \bar{\xi}$ denote partial derivatives with respect to these variables. In terms of the potential $\mathcal{E}(\xi, \bar{\xi})$, Einstein's equations for the metric (1) in particular imply the Ernst equation [5]

$$[(\xi - \bar{\xi})g_\xi g^{-1}]_{\bar{\xi}} + [(\xi - \bar{\xi})g_{\bar{\xi}} g^{-1}]_\xi = 0, \quad (3)$$

with the symmetric matrix

$$g = \frac{1}{\mathcal{E} + \bar{\mathcal{E}}} \begin{pmatrix} 2 & i(\mathcal{E} - \bar{\mathcal{E}}) \\ i(\mathcal{E} - \bar{\mathcal{E}}) & 2\mathcal{E}\bar{\mathcal{E}} \end{pmatrix}. \quad (4)$$

Equation (2) for $k(\xi, \bar{\xi})$ may be equivalently written in the form

$$d \ln h = q, \quad (5)$$

where $h \equiv e^{2k}$ is the conformal factor. Using (3) one can show that the one-form q defined by

$$q = \frac{\xi - \bar{\xi}}{4} \text{tr}(g_\xi g^{-1})^2 d\xi + \frac{\bar{\xi} - \xi}{4} \text{tr}(g_{\bar{\xi}} g^{-1})^2 d\bar{\xi} \quad (6)$$

is closed, i.e., $dq = 0$. Equation (3) is the compatibility condition of the following linear system [6,7]:

$$\frac{d\Psi}{d\xi} = U\Psi, \quad \frac{d\Psi}{d\bar{\xi}} = V\Psi, \quad (7)$$

where

$$U = \frac{g_\xi g^{-1}}{1 - \gamma}, \quad V = \frac{g_{\bar{\xi}} g^{-1}}{1 - \gamma}; \quad (8)$$

and $\Psi(\gamma, \xi, \bar{\xi})$ is a 2×2 matrix, from which the Ernst potential and thus the metric (1) can be reconstructed. The function $\gamma(\xi, \bar{\xi})$ is a “variable spectral parameter” subject to the following (compatible) first order equations:

$$\gamma_\xi = \frac{\gamma}{\xi - \bar{\xi}} \frac{1 + \gamma}{1 - \gamma}, \quad \gamma_{\bar{\xi}} = \frac{\gamma}{\bar{\xi} - \xi} \frac{1 - \gamma}{1 + \gamma}. \quad (9)$$

They are solved by

$$\gamma(\xi, \bar{\xi}, w) = \frac{2}{\xi - \bar{\xi}} \left\{ w - \frac{\xi + \bar{\xi}}{2} - \sqrt{(w - \xi)(w - \bar{\xi})} \right\}, \quad (10)$$

with $w \in \mathbf{C}$ a constant of integration, which can be regarded as the “hidden” constant spectral parameter. For the linear system (7), we can use either γ or w ; when γ is expressed as a function of w according to (10), the linear system (7) lives on the two-sheeted Riemann surface of the function $\sqrt{(w - \xi)(w - \bar{\xi})}$. Both the constant and the variable spectral parameters w and γ are needed for a

proper understanding of the (infinite-dimensional) hidden symmetries of (7) [8,9]. Furthermore,

$$\begin{aligned} \frac{d}{d\xi} &\equiv \frac{\partial}{\partial\xi} + \frac{\gamma}{\xi - \bar{\xi}} \frac{1 + \gamma}{1 - \gamma} \frac{\partial}{\partial\gamma}, \\ \frac{d}{d\bar{\xi}} &\equiv \frac{\partial}{\partial\bar{\xi}} + \frac{\gamma}{\bar{\xi} - \xi} \frac{1 - \gamma}{1 + \gamma} \frac{\partial}{\partial\gamma}. \end{aligned} \quad (11)$$

The poles of (7) in the complex γ plane are thus produced by differentiation of γ according to (9).

Choosing $(\gamma, \xi, \bar{\xi})$ as independent variables, we get the following relations from (7) and (9) [10]:

$$\begin{aligned} g_\xi g^{-1} &= \frac{2}{\xi - \bar{\xi}} \Psi_\gamma \Psi^{-1}|_{\gamma=1}, \\ g_{\bar{\xi}} g^{-1} &= \frac{2}{\bar{\xi} - \xi} \Psi_\gamma \Psi^{-1}|_{\gamma=-1}, \end{aligned} \quad (12)$$

where the subscript γ denotes differentiation with respect to γ .

Deformation equations.—Although we shall keep in mind the Ernst equation and its associated linear system (7) as our principal example, the results described below hold for arbitrary $GL(n, \mathbb{C})$ -valued matrices $g(\xi, \bar{\xi})$, as well as for the gravitationally coupled nonlinear σ models obtained by dimensional reduction of Maxwell-Einstein theories in higher dimensions. For our analysis, we shall use the general framework of monodromy preserving deformations of ordinary differential equations [1].

Let us now consider the behavior of $(d\Psi/d\xi)\Psi^{-1}$ and $(d\Psi/d\bar{\xi})\Psi^{-1}$ in the complex γ plane. Singularities in γ arise at those points where $\Psi(\gamma)$ is either nonholomorphic or degenerate (i.e., $\det \Psi = 0$). Analyticity away from the points $\gamma = \pm 1$ implies that all singular points γ_j (for $j = 1, \dots, N$) of the function $\Psi(\gamma, \xi, \bar{\xi})$ are regular in the sense that [1]

$$\Psi(\gamma) = G_j(\xi, \bar{\xi}) \Psi_j(\gamma, \xi, \bar{\xi}) (\gamma - \gamma_j)^{T_j} C_j \quad \text{as } \gamma \sim \gamma_j. \quad (13)$$

For $\gamma \sim \gamma_j$, $\Psi_j(\gamma, \xi, \bar{\xi}) = 1 + O(\gamma - \gamma_j)$ is holomorphic and invertible. The matrices C_j and T_j are constant and invertible, and constant diagonal, respectively, while the $(\xi, \bar{\xi})$ -dependent matrices G_j are assumed to be invertible. The singular points γ_j depend on $(\xi, \bar{\xi})$ according to (9), i.e., we have $\gamma_j = \gamma(w_j, \xi, \bar{\xi})$ with constants $w_j \in \mathbb{C}$ [11]. The set $\{\gamma_j, C_j, T_j\}$ for $j = 1, \dots, N$ is generally referred to as the set of monodromy data of $\Psi(\gamma)$. The function $\Psi(\gamma)$ is uniquely defined by its monodromy data up to normalization [1].

The logarithmic derivative $\Psi_\gamma \Psi^{-1}$ is thus holomorphic except at the points $\gamma = \gamma_j$ where it has simple poles with residues

$$A_j(\xi, \bar{\xi}) = G_j T_j G_j^{-1} \quad (14)$$

by (13). The functions $A_j(\xi, \bar{\xi})$ will play a central role in the sequel. In general the number N of regular singularities γ_j may be infinite (explicit examples are the x -periodic static axisymmetric solutions found in [12]) or

even continuous (this would correspond to nonconstant conjugation matrices in the related Riemann-Hilbert problem). However, in this paper we will restrict attention to finite N . Besides that, we find it convenient to impose the normalization condition $\Psi_\gamma \Psi^{-1}|_{\gamma=\infty} = 0$, which may be ensured, for instance, by demanding $\Psi|_{\gamma=\infty} = \sigma_3$. A large class of solutions with finitely many singularities is provided by the multisoliton solutions of Einstein's equations in [7] (corresponding to matrices T_j , all of whose eigenvalues are half-integer) and the finite-gap (algebro-geometric) solutions constructed in [13].

Combining (13) and (14) we arrive at the following differential equation in γ :

$$\frac{\partial \Psi}{\partial \gamma} = \sum_{j=1}^N \frac{A_j}{\gamma - \gamma_j} \Psi. \quad (15)$$

Inserting (15) into (12), we immediately obtain

$$\begin{aligned} g_\xi g^{-1} &= \frac{2}{\xi - \bar{\xi}} \sum_j \frac{A_j}{1 - \gamma_j}, \\ g_{\bar{\xi}} g^{-1} &= \frac{2}{\bar{\xi} - \xi} \sum_j \frac{A_j}{1 + \gamma_j} \end{aligned} \quad (16)$$

(in the sequel summation is taken everywhere from 1 to N). Substituting (16) into (7), we get the following compatibility conditions between (7) and (15) [14]:

$$\begin{aligned} \frac{\partial A_j}{\partial \xi} &= \frac{2}{\xi - \bar{\xi}} \sum_{k \neq j} \frac{[A_k, A_j]}{(1 - \gamma_k)(1 - \gamma_j)}, \\ \frac{\partial A_j}{\partial \bar{\xi}} &= \frac{2}{\bar{\xi} - \xi} \sum_{k \neq j} \frac{[A_k, A_j]}{(1 + \gamma_k)(1 + \gamma_j)} \quad (j = 1, \dots, N). \end{aligned} \quad (17)$$

It is now straightforward to check that this system is always compatible if the functions γ_j obey (9). The first main result of this Letter is the following.

Theorem 1: *Let $\{w_j \in \mathbb{C}; j = 1, \dots, N\}$ be an arbitrary set of complex constants and $A_j = A_j(\xi, \bar{\xi})$ an associated set of solutions of (17). Then the system of linear equations (16) is always compatible, and the matrix function $g = g(\xi, \bar{\xi})$ obtained by integrating (16) solves (3).*

The proof may be obtained by direct calculation.

It is quite remarkable that the dependence of the Ernst equation and its associated linear system on the variables ξ and $\bar{\xi}$ can be completely decoupled by Theorem 1. In other words, the problem of solving Einstein's equations in this reduction has been reduced to integrating two ordinary matrix differential equations, which are automatically compatible, unlike to original linear system (7). All information about the degrees of freedom is thereby encoded into the "initial values," i.e., the set of (constant) matrices $A_j^{(0)} \equiv A_j(\xi^{(0)}, \bar{\xi}^{(0)})$; these are also the appropriate phase space variables, as we will see below. Accordingly, we will regard the functions $A_j(\xi, \bar{\xi})$ rather than $\Psi(\gamma, \xi, \bar{\xi})$ as the fundamental quantities from now on, and relate the system (17) directly to the (complexified) Ernst equation (3).

Equations (17) may also be represented in “Lax form,” viz.,

$$\frac{\partial A_j}{\partial \xi} = [U|_{\gamma=\gamma_j}, A_j], \quad \frac{\partial A_j}{\partial \bar{\xi}} = [V|_{\gamma=\gamma_j}, A_j], \quad (18)$$

where the matrices U and V are defined in (8) and (16). This form of (17) is “gauge covariant” with respect to the transformation

$$\tilde{\Psi} = \Omega(\xi, \bar{\xi})\Psi. \quad (19)$$

Namely, the transformed function $\tilde{\Psi}$ satisfies the linear system $d\tilde{\Psi}/d\xi = \tilde{U}\tilde{\Psi}$, $d\tilde{\Psi}/d\bar{\xi} = \tilde{V}\tilde{\Psi}$, where

$$\tilde{U} = \Omega_\xi \Omega^{-1} + \Omega U \Omega^{-1}, \quad \tilde{V} = \Omega_{\bar{\xi}} \Omega^{-1} + \Omega V \Omega^{-1}. \quad (20)$$

Clearly, the matrix functions A_j transform as $A_j \rightarrow \tilde{A}_j = \Omega A_j \Omega^{-1}$ under (19). The transformed matrices \tilde{A}_j then obey the same linear system (18) with the pair (U, V) replaced by (\tilde{U}, \tilde{V}) .

Theorem 1 establishes a direct correspondence between $GL(2, \mathbb{C})$ -valued solutions of (3) and solutions of (17). However, it does not specify the conditions that must be imposed on $\{w_j, A_j\}$ in order to satisfy certain restrictions which the metric g may be subject to. It is easy to see that the condition $\det g = 1$ is guaranteed by $\text{tr} A_j = 0$; reality of g requires the existence of an involution (complex conjugation) on the set $\{w_j, A_j\}$. Conditions that ensure $g = g^T$ are more difficult to formulate and will be discussed elsewhere.

Conformal factor and τ function.—To each solution $\{A_j\}$ of (17) we can associate the following closed one-form [1]:

$$q_0(\xi, \bar{\xi}) = \sum_{j \neq k} \text{tr}(A_j A_k) d \ln(\gamma_j - \gamma_k), \quad (21)$$

where the exterior derivative d is to be taken with respect to the deformation parameters $(\xi, \bar{\xi})$. The closure condition $dq_0 = 0$ may be directly verified by use of (17) and (9). Following the general prescription given in [1], we define the τ function associated with the Ernst equation by

$$d \ln \tau = q_0. \quad (22)$$

We will now show that this τ function has a very definite physical meaning in our context: Up to an explicit factor, it is just the conformal factor $h \equiv e^{2k}$ [15]. To establish this result, we first substitute (16) into (6); then using (9) and (21) we obtain

$$q = q_0 + \frac{1}{\xi - \bar{\xi}} \sum_j \text{tr} A_j^2 \left[\frac{d\xi}{(1 - \gamma_j)^2} - \frac{d\bar{\xi}}{(1 + \gamma_j)^2} \right] + \sum_{j < k} \text{tr}(A_j A_k) d \ln(\xi - \bar{\xi}). \quad (23)$$

Now, from (17) it follows that $\sum_j A_j$ is $(\xi, \bar{\xi})$ independent. Furthermore, it is easy to check that $\text{tr} A_j$ and $\text{tr} A_j^2$ are independent of ξ and $\bar{\xi}$, hence constant, for all j . Therefore, the eigenvalues of A_j are $(\xi, \bar{\xi})$ independent

for any solution of (17) that agrees with (14). As a consequence, the expression $\sum_{j < k} \text{tr}(A_j A_k)$ is likewise $(\xi, \bar{\xi})$ independent, and all extra terms on the right-hand side (r.h.s.) of (23) may be explicitly integrated.

Using (9) and (10), we thus arrive at the following.

Theorem 2: The conformal factor h (5) and the τ function (22) are related by

$$h(\xi, \bar{\xi}, w_j) = C(\xi - \bar{\xi})^{1/2 \text{tr} \{ \sum_j A_j \}^2} \times \prod_j \left[\frac{\partial \gamma_j}{\partial w_j} \right]^{(1/2) \text{tr} A_j^2} \tau(\xi, \bar{\xi}, w_j), \quad (24)$$

where $C \in \mathbb{C}$ is a constant of integration.

Notice once more that quantities $\text{tr}(\sum_j A_j)^2$ and $\text{tr} A_j^2$ are $(\xi, \bar{\xi})$ independent. If the related matrix g is real and symmetric, then $\sum_j A_j = 0$, and the first factor on the r.h.s. of (24) drops out. We emphasize that our result is more general than previous ones (the explicit computability of h for multi-soliton solutions has been known for a long time [7]), and valid for arbitrary nonlinear σ models coupled to gravity.

Hamiltonian formulation.—The system (17) is a “two-time” Hamiltonian system with respect to the standard Lie-Poisson bracket [3,16]

$$\{A(\gamma) \otimes A(\mu)\} = [r(\mu - \gamma), A(\gamma) \otimes \mathbf{1} + \mathbf{1} \otimes A(\mu)], \quad (25)$$

where $A(\gamma) \equiv \Psi_\gamma \Psi^{-1}$ and the classical rational R matrix $r(\gamma)$ is equal to Π/γ with Π the permutation operator in $\mathbb{C}^2 \times \mathbb{C}^2$. The dynamics in the ξ and $\bar{\xi}$ directions are governed by the Hamiltonians

$$H_1 \equiv 2k_\xi = \frac{1}{\xi - \bar{\xi}} \sum_{k,j} \frac{\text{tr}(A_j A_k)}{(1 - \gamma_j)(1 - \gamma_k)},$$

$$H_2 \equiv 2k_{\bar{\xi}} = \frac{1}{\bar{\xi} - \xi} \sum_{k,j} \frac{\text{tr}(A_j A_k)}{(1 + \gamma_j)(1 + \gamma_k)}. \quad (26)$$

Compatibility of the system (17) implies $\{H_1, H_2\} = 0$, i.e., the flows with respect to the two “time variables” ξ and $\bar{\xi}$ commute (as can also be verified by explicit computation). Note that our formulation is far simpler both technically and conceptually than previous Hamiltonian treatments of such systems, which were based on the use of “one-time” Hamiltonians, and where the Lie-Poisson brackets (25) would, in addition, depend on the space coordinates. The notorious problems caused by derivatives of δ functions (so-called “nonultralocal” terms) in the relevant Poisson brackets are altogether avoided here. Furthermore, the cumbersome structure of the canonical current algebra in the conventional approach is replaced by a more transparent algebraic structure in our framework.

To be sure, we should regard (26) as constraints à la Dirac rather than conventional Hamiltonians, because (3) is derived from a *generally covariant* theory. To do this properly would, however, require that we undo the

choice of Weyl coordinates, on which (1) and (3) are based, and to treat ξ and $\bar{\xi}$ as canonical variables subject to $\{2k_\xi, \xi\} = \{2k_{\bar{\xi}}, \bar{\xi}\} = 1$ and $\{2k_\xi, \bar{\xi}\} = \{2k_{\bar{\xi}}, \xi\} = 0$. The quantities $\Phi_1 \equiv 2k_\xi - H_1$ and $\Phi_2 \equiv 2k_{\bar{\xi}} - H_2$ are thereby converted into (mutually commuting) constraint operators on an enlarged phase space (they are, in fact, just the Virasoro constraints). The “time evolutions” of the spectral parameter γ are also generated canonically in the sense that $\gamma_\xi = \{\Phi_1, \gamma\}$ and $\gamma_{\bar{\xi}} = \{\Phi_2, \gamma\}$ (so “time” must eventually be quantized in this scheme).

We close with the following final remarks:

(1) An obvious advantage of using the variables A_j in comparison with the ones employed traditionally in this context is that they generate a closed Lie algebra with respect to the standard Lie-Poisson bracket (25). Secondly, the common features of our system (17) with the classical limit of the Knizhnik-Zamolodchikov (KZ) equations [16,17] suggest that one should quantize (17) in analogy with the KZ equations (although quantum gravity will certainly introduce new features). We have only discussed the case of finite N in this paper, but there are in principle no obstacles to considering $N = \infty$ from the outset, since the space of finite N solutions can be naturally embedded in this larger space. For illustrative purposes, it is quite useful to think of the set of finite N solutions as the “ N -particle sector” of the theory because, depending on the reality conditions, the solutions g corresponding to (15) possess exactly N or $N/2$ singularities w_1, \dots, w_N in the upper ξ half plane. In fact, a proper treatment of the Ernst equation as a (generally covariant) quantum field theory will presumably require taking into account N as a “particle number operator.”

(2) The extension of our results to the case of a Lorentzian world sheet and to arbitrary G/H coset space σ models coupled to gravity in two space-time dimensions is straightforward. In the notation of [9], where this case is reviewed in some detail, the linear system matrix Ψ corresponds to $\hat{V}\eta(\mathcal{V})^{-1}$, where η denotes the Cartan involution [e.g., $\eta(\mathcal{V}) = (\mathcal{V}^T)^{-1}$ for $G = \text{SL}(n)$]; furthermore, the spectral parameter t used there corresponds to i times the parameter γ employed in the present paper. For arbitrary coset spaces, the matrices A_j belong to the Lie algebra of G ; like g , they may be subject to further restrictions.

(3) Analogs of the static axisymmetric (multi-Schwarzschild) solutions for arbitrary σ models can be easily constructed in our formalism by choosing the matrices A_j in the Cartan subalgebra of the relevant Lie algebra. From (17) it is then immediately evident that $A_j = \text{const}$.

(4) Obviously our formulation will yield a new realization of the Geroch group [18] and its generalizations; we here just note that this group mixes sectors belonging to different “particle numbers.” It is known that the corre-

sponding Kac-Moody algebras act on the conformal factor via their central extension [8,9,19]; combining this result with our Theorem 2 should shed some light on the group theoretical meaning of the τ function.

A detailed account of the results described in this Letter is in preparation.

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