Breakdown of Hydrodynamics in a One-Dimensional System of Inelastic Particles

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We study dynamics of nearly elastic particles constrained to move on a line with energy input from the boundaries. We find that for typical initial conditions, the system evolves to an "extraordinary" state with particles separated to two groups: The majority of the particles get clamped into a small region of space and move with very slow velocities; a few remaining particles travel between the boundaries at much higher speeds. Such a state clearly violates equipartition of energy. The simplest hydrodynamic approach fails to give a correct description of the system.

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In the study of a many-particle system with interactions, a hydrodynamic approach is naturally used when the quantity of interest is of a macroscopic nature with length and time scale much larger than the typical microscopic scales. Instead of focusing on the detailed dynamics of individual particles, one usually writes down a set of equations that describe the evolution of some macroscopic quantities such as local density, temperature, and flow velocity, with transport coefficients derived from statistical considerations [1]. The derivation of the hydrodynamic equations [2] relies on the assumptions that the particles reach local equilibrium with equipartition of energy, and the statistical properties can be simply described by local macroscopic quantities, i.e., the temperature. However, as we shall demonstrate that, with the introduction of a very small dissipation in the microscopic dynamics, a many-particle system can reach "extraordinary" states where local equilibrium is destroyed. As this happens, the hydrodynamic approach fails to give a correct picture for the system.

The specific example we shall show in this Letter is a one-dimensional many-particle system in which particles interact via *inelastic* collisions which conserve momentum but dissipate kinetic energy. Such a model was originally motivated by the studies of granular materials [3–7]. Consider a horizontal column of N sizeless inelastic particles with identical mass confined by two walls of infinite masses $L = 1$ apart. When two particles collide, the velocities after collision, v_1' and v_2', are expressed in terms of the velocities before collision, v_1 and v_2, as

$$\begin{aligned} v_1' &= \epsilon v_1 + (1 - \epsilon) v_2 , \\ v_2' &= (1 - \epsilon) v_1 + \epsilon v_2 . \end{aligned} \tag{1}$$

Here $\epsilon = (1 - r)/2$, with r the coefficient of restitution defined by $v_1' - v_2' = -r(v_1 - v_2)$. If $r = 1$, the collision is perfectly elastic, and if $r = 0$, the collision is completely sticky. If there is no energy input, all the particles will come to rest after the initial kinetic energy is dissipated through collisions. To see nontrivial dynamics, one has to drive the system. We choose to drive the system by pumping energy from the left side wall. The rule is as follows. When the leftmost particle hits the left wall, it will be returned with a random velocity v_0 of a Gaussian distribution, $\sim \exp(-v_0^2/2T_0)$. The collision between the rightmost particle and right wall is perfectly elastic, resulting in no energy change. The above boundary conditions therefore mimic a left wall held at constant temperature and a right wall thermally insulated in a thermodynamic sense.

In a conventional hydrodynamic approach, one describes the system by a set of macroscopic quantities: particle number density $\rho(x,t) \equiv \langle \sum_i \delta(x - x_i(t)) \rangle$, macroscopic flow velocity $u(x,t) \equiv \langle \sum_i v_i(t) \delta[x_i(t) - x] \rangle$, and temperature $T(x,t) \equiv \langle \sum_i [v_i(t) - u(x,t)]^2 \delta(x_i(t) - x) \rangle$, where the $\langle\ \rangle$ represents a coarse-grained average over a small region of space or time. Assuming local equilibrium, one can derive the following set of equations based on mass, momentum, and energy balances [8,9]:

$$\begin{aligned} \partial_t \rho &= -\partial_x(\rho u) , \\ \rho \partial_t u &= -\rho u \partial_x u - \partial_x(C_1 \rho T) , \\ \rho \partial_t T &= -\rho u \partial_x T - C_1 \rho T \partial_x u + \partial_x^2 (C_2 T^{3/2}) - C_3 \epsilon \rho^2 T^{3/2} , \end{aligned} \tag{2}$$

with C_1, C_2, and C_3 numerical constants. The above equations were derived in the context of granular sand flow. They are similar to the Navier-Stokes equation for an ideal gas except that inelastic collisions between particles lead to an additional energy dissipation term $C_3 \epsilon \rho^2 T^{3/2}$ in the energy balance equation.

What do the hydrodynamic equations tell us about our system? Given the boundary conditions $T(0,t) = T_0$, $\partial T/\partial x(1,t) = 0$, and $\rho u = 0$ at $x = 0, 1$ (no mass flow through the walls), the above equations predict that, regardless of the initial condition, the final state is a steady state with no macroscopic flow. This steady state can be solved analytically. The solution is smooth and stable (see Fig. 1 for a typical solution). We note that as $\epsilon \to 0$ with N fixed, $T(x)$ and $\rho(x)$ become uniform.

We now contrast the prediction of the hydrodynamic equations with our numerical simulation. We simulate this system by using an event-driven code which searches out where and when the next collision happens [5,6,10,11]. The configuration is updated after each collision. We use a number of particles N and a coefficient of restitution r such that $N(1 - r) < 1$, while N is large.

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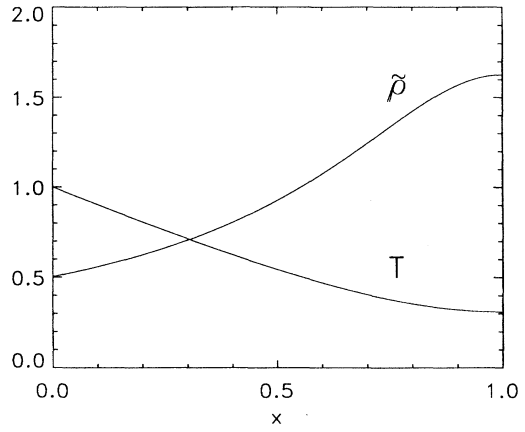

FIG. 1. Density ($\tilde{\rho} = \rho/N$) and temperature (T) profiles from the steady state solution of the hydrodynamic equations. Here $N = 100$, $\epsilon = 0.005$, and $C_2/C_3 = 11.8$.

Such a restriction is necessary since it is known [5,12] that, for sufficiently large $N(1 - r)$, one encounters a singularity where a group of particles become so sticky that they collide with each other infinitely often in a finite amount of time.

We start the simulation by generating an arbitrary initial configuration. Figure 2 shows the result of a typical simulation. At the beginning the particles are uniformly distributed. Only the leftmost particle has a nonzero velocity. Soon all the particles except the leftmost one move toward the right wall. This motion continues until all the $N - 1$ particles are squeezed at the right wall and get "clamped" into a small region. The size of the clamped cluster is very small compared to the dimension of the box, and the particles in the cluster move with a very small velocity. The leftmost particle travels at a relatively much faster speed, $v \sim \sqrt{T_0} = 1$, between the left wall and the cluster, delivering the energy it gained from the collision with the left wall to the cluster. The collisions between the fast particle and the cluster also provide the necessary pressure to keep the cluster clamped. When the leftmost particle obtains from the left side wall a velocity far less than its typical velocity, $\sqrt{T_0} = 1$, the cluster then may have time to burst out [Fig. 2(d)], and after a transient time, the $N - 1$ particles will again get clamped at the right wall [Fig. 2(e)]. Figure 3 shows the motion of the center of mass. In this case, the typical size of the cluster is $\sim 10^{-3}$ and the typical velocity of the particles in the cluster is $\sim 10^{-3}$. Notice that there are frequent bursts of the cluster when the velocity of the incident particle is very small (due to the random Gaussian distribution). These bursts are then brought back to the clamped states after tens of collisions with the fast particle, indicating that the $N - 1$ particles tend to stick to the right wall. We have checked various initial conditions to make sure that the realization of this final state is independent of the initial condition. For example, we tried an initial condition with particles having randomly chosen velocity between $-\sqrt{T_0}$ and $\sqrt{T_0}$.

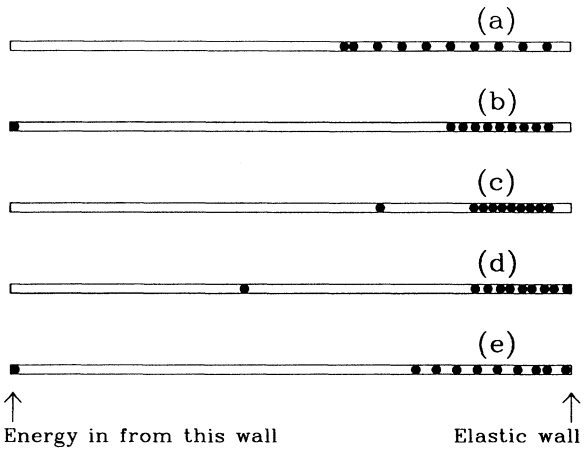

FIG. 2. Snapshots of a ten-particle system. Here $\epsilon = 0.05$ and $T_0 = 1$. (a) Snapshots of particles at a transient time. (b), (c) The "extraordinary state" where nine particles are squeezed into a small space an one particle runs fast between these nine particles and the left side walls. (d) A bursting cluster. (e) The burst is brought back to the clamped state. Note that in order to make particles visible, we assume each particle is a sphere of radius $r = 0.01$, thus the plotted particles center at $x' = x + (2i - 1)r$, where i is the particle index counting from the left to the right, i.e., the leftmost particle has an index $i = 1$.

We observe that the particles collide with each other and the initial kinetic energy gets dissipated after a transient period so that eventually the system ends up to the same final state as described above. We have simulated systems with various N and ϵ (with $N\epsilon < 1$), the above picture does not change. We should emphasize that the formation of the clump does not disappear as $\epsilon \to 0$, contrary to the prediction of the hydrodynamic equations. In fact, with fixed N and decreasing ϵ, the particles in the clump get squeezed into a smaller space and move with slower speeds.

Note that if we replace the inelastic particles by perfectly elastic ones, the situation is completely different. The

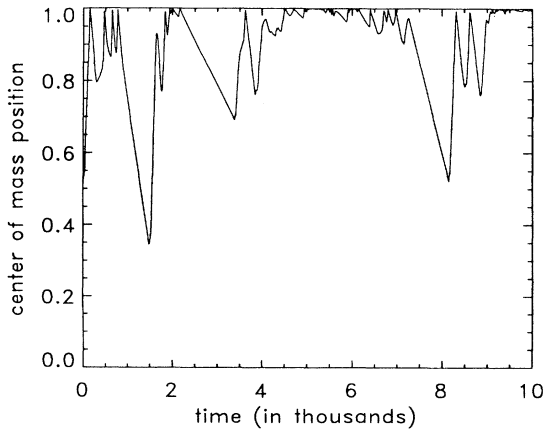

FIG. 3. Position of center of mass versus time for $N = 100$, $\epsilon = 0.005$, and $T_0 = 1$.

collision between two particles simply causes them to exchange their velocities. After a while, the initial velocity of any particle will be given to the leftmost one and get randomalized by the collision with the wall. Thus, the system will eventually reach a state where the particles distribute themselves uniformly in space, with velocities assuming the same Gaussian distribution as v_0. Therefore, the inelasticity plays an important role in destroying the local equilibrium.

The above inelastic result is independent of how the energy is pumped in at the boundary. For example, similar phenomenon occur when the left wall is rapidly oscillating with a small amplitude or when the wall simply kicks the leftmost particle back with a constant velocity. (In the latter case, we note that the number of fast particles between the left wall and the "clump" depends on the initial condition.) We also simulated a case where both side walls kick back the incoming particle with a constant velocity. In this case, almost all particles are squeezed to form a cluster and move slowly between the two side walls with two groups of fast particles running between the cluster and two side walls.

The case where the left wall kicks the particle back with constant velocity $v_0 = 1$ and initially $N - 1$ rightmost particles are at rest is quite illustrative, since the final state is periodic and detailed dynamics can be well understood. In this case, $N - 1$ particles are squeezed into a very small cluster, colliding with the fast particle and the wall as if they were just one big particle. Figure 4 plots the center of mass motion of the cluster as a function of time. We see that initially the center of mass accelerates toward the right wall until the cluster hits the wall with a large velocity and bounces back. After several collisions with the right wall and the leftmost particle, the cluster begins a periodic motion. In one period, the cluster first moves with a constant velocity V_{in} towards the wall. After colliding with the wall, it comes back with a constant velocity V_{out}. This velocity is changed back to V_{in} after the collision with the fast particle. This process repeats with the period equal to the traveling time of the fast particle.

The above state can be understood analytically by using the following simple picture. Consider $N - 1$ particles initially placed close to the wall with zero velocity, label them in order as particle $1, 2, \ldots, N - 1$ with particle 1 closest to the wall. Let particle N incident in from left with unit velocity. Since the collision is nearly elastic, after first collision between particle N and $N - 1$, particle $N - 1$ acquires a large velocity $1 - \epsilon$, and particle N nearly comes to rest with a small velocity ϵ. Then the next collision will be between particle $N - 1$ and $N - 2$, so on so forth, until particle 2 collides with particle 1, giving particle 1 a velocity $(1 - \epsilon)^{N-1}$. This velocity is reversed after the elastic collision of particle 1 with the wall, and the collisions propagate back to particle N and knock it out with a velocity $(1 - \epsilon)^{2(N-1)}$. This completes the collision of fast particle with the cluster, giving the cluster a center of mass velocity, $V_d = [1 + (1 - \epsilon)^{2(N-1)} - 2(1 - \epsilon)^{N-1}]/(N - 1) \approx [1 - \exp(-N\epsilon)]^2/(N - 1)$, in the limit $N \gg 1$ and $\epsilon \gg 1$. Therefore, the cluster acquires a small drifting velocity V_d towards the wall. For a small $N\epsilon$, $V_d \sim N\epsilon^2$.

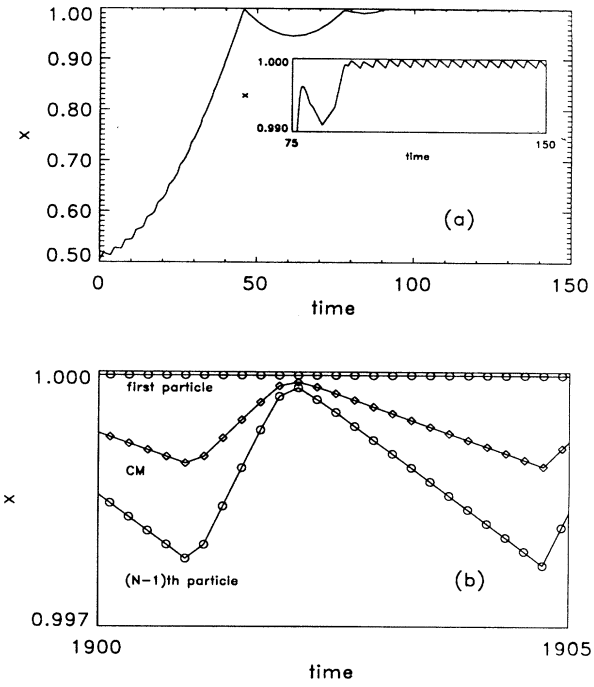

FIG. 4. Position of center of mass versus time for $N = 100$, $\epsilon = 0.005$. The left side wall kicks the incident particle with a velocity $v_0 = 1$. (a) The transient period. The inset is a blowup showing that after the transient, the c.m. moves periodically forever. (b) The periodic motion of the center of mass for one period. In (b) we also plot the positions of the first (closest to the right wall) and $(N - 1)$th particles to show that the clump synchronously undergoes periodic size changes. The plotted solid lines in (b) simply connect data. From (b), we obtain $V_{\text{in}} = 1.11 \times 10^{-3}$ and $V_{\text{out}} = 4.08 \times 10^{-4}$.

To figure out how the center of mass velocity of the drifting cluster is changed due to collisions with the wall, consider simply that the $N - 1$ particles move with some velocity V_{in} towards the wall. First, particle 1 reverses its velocity by a collision with the wall, then particle 1 collides with 2. The collision propagates until particle $N - 2$ collides with $N - 1$, so that $N - 1$ gets a velocity away from the wall. The same sequence of collision repeats itself, starting with a collision between particle 1 and the wall. After $N - 1$ such collision wave, the velocities of all the $N - 1$ particles change direction, and the whole cluster moves away from the wall with a center of mass velocity $V_{\text{out}} = \alpha V_{\text{in}}$, where α is a constant which depends on N and ϵ. For $N\epsilon \ll 1$, $\alpha = 1 - 2(N - 2)\epsilon$. There are another $N - 1$ waves of collisions between the particles (not with the wall) which rearrange the velocities (but do not affect V_{out}). We find after the rearrangement the velocity differences between the consecutive particles are constant, $\Delta V/V_{\text{in}} \sim 2(N - 3)\epsilon^2$ to lowest order, leading to a uniformly expanding column.

We now use the above collision pictures to calculate several quantities in the final state. Since $V_{\text{in}} = V_d - V_{\text{out}}$ and $\alpha V_{\text{in}} = V_{\text{out}}$, we obtain $V_{\text{in}} = V_d/(1+\alpha)$ and $V_{\text{out}} = \alpha V_d/(1+\alpha)$. The width of the cluster l can be roughly estimated by the distance the center of mass moves away from the wall, giving $l \approx \alpha V_d/(1+\alpha)^2$, assuming the traveling period of the fast particle is 1. For a small $N\epsilon$, $l \sim N\epsilon^2$, and $V_{\text{in}} \approx V_{\text{out}} \sim N\epsilon^2$. This analysis shows that with N fixed and $\epsilon \to 0$, the clamped state become even singular. For $N = 100$ and $\epsilon = 0.005$, we find $\alpha = 0.34$, $V_{\text{in}} = 1.17 \times 10^{-3}$ and $V_{\text{out}} = 3.97 \times 10^{-4}$, which agrees quite well with the simulation. Numerically, we also find that the cluster is uniformly expanding while it is moving away from the wall.

The above scalings can also be understood via a Boltzmann equation for one-dimensional inelastic collisions [13]. Let $f(v,x,t)$ be the phase-space distribution density function such that $f(v,x,t)dxdv$ is the number of particles located between x and $x+dx$ with velocities between v and $v+dv$. Then the Boltzmann equation for this system is

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f(v,x,t) = -\int du\,dU\,dV |u-v| f(v,x,t) f(u,x,t) \delta(V - v + \epsilon(v-u)) \delta(U - u + \epsilon(u-v)) + \int du\,dU\,dV |U-V| f(V,x,t) f(U,x,t) \delta(v - V + \epsilon(V-U)) \delta(u - U + \epsilon(U-V)), \quad (3)$$

where the two terms on the right hand side are the usual scattering out and scattering into beam terms of the Boltzmann equation. For small ϵ, the equation can be simplified as

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) f(v,x,t) = \epsilon \frac{\partial}{\partial v} \int du\,(v-u) f(v,x,t) f(u,x,t) |v-u|. \quad (4)$$

When $N \to \infty$ and $\epsilon \to 0$ with $N\epsilon$ fixed, the $N-1$ particles clumped into a small region of space $\sim N\epsilon^2$ and moved at a typical velocity $\sim N\epsilon^2$. We then have that $f \sim N/(N\epsilon^2)^2$. The two left hand side terms are $\sim 1/N\epsilon^4$, which are balanced by the right hand side term which is $\sim N\epsilon/(N\epsilon^4)$. Note that this approach implies that in the limit $\epsilon \to 0$ with $N\epsilon =$ const the system converges to a well-defined distribution function.

We now give a rough estimate of the transient time. For a system with no energy input, it takes certain amount of time to dissipate its initial energy, call such a time τ_d. Consider also a system driven by the boundary energy input, with an initial condition that all the particles are uniformly distributed with negligible velocity. Call τ_s the time it takes to squeeze the particles to the end. We assume that the transient time is the larger of τ_d and $\tau_s \cdot \tau_s$ can be estimated using the drift velocity the cluster acquired after each collision with the fast particle. This leads to $\int_0^{\tau_s} t\,dt\,V_d \sim 1$, which yields $\tau_s \sim \sqrt{2/V_d}$. For a system where particles have random initial velocities V_0 and uniform density n, a simple energy balance leads to $\tau_d \sim 1/\epsilon n V_f$, where V_f is the typical velocity of clamped particles. For $N = 100$ and $\epsilon = 0.005$, we find $\tau_s \sim 36$, which roughly agrees with the simulation that starts with particles having zero initial velocities.

A similar phenomenon has been previously reported in the cooling of inelastic particles [12–14]. In that case, a clump forms as particles gradually lose their kinetic energy. Here we have seen that, driven by boundary energy sources, a one-dimensional system of many inelastic particles may as well collapse into an extraordinary state where particles move with velocities of totally different magnitudes (i.e., no equipartition of energy) and therefore hydrodynamics fails to give a correct description. It remains to be seen whether such a behavior persists in a driven system in higher dimensions.

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