Exact Solution of the Sutherland Model with Arbitrary Internal Symmetry

Yusuke Kato and Yoshio Kuramoto
Department of Physics, Tohoku University, Sendai 980, Japan
(Received 8 September 1994)

An elementary theory is presented for solving the Sutherland model with arbitrary internal symmetry such as SU(ν) or a supersymmetry SU(ν, μ). The ground-state wave function and all the energy levels are derived. One starts with solving a variant of the model with distinguishable particles, and then (anti)symmetrizes the solution. The theory is also applied to various lattice versions of the model. It is proved that the Gutzwiller-type wave function is not only an eigenstate of the supersymmetric *t-J* model, but is indeed the ground state.

PACS numbers: 75.10.Jm, 05.30.–d

There has been an increasing interest in a family of one-dimensional models with interaction proportional to the inverse square of the distance. With the periodic boundary condition, the model is called the Sutherland model [1]. Several years ago, Haldane [2] and Shastry [3] found that an $S = 1/2$ spin chain with long-ranged exchange (HS model) is solvable and that it is a lattice version of the original Sutherland model. After their work, the charge degrees of freedom, the *hole,* was introduced into the HS model and the long-ranged supersymmetric *t-J* model has been proposed by Kuramoto and Yokoyama [4]. Since then, the Sutherland model, HS model, and *t-J* model have been generalized to multicomponent systems and the generalized models have been investigated intensively [5–7]. The connection between the degeneracy of the spectrum and the Yangian symmetry was discussed [8,9]. In view of the fundamental physical interest in the exact spin and charge dynamics in the *t-J* model, for instance, it is highly desirable to develop a simple and flexible theoretical scheme for systems with internal symmetries.

Sutherland and Shastry applied the asymptotic Bethe ansatz (ABA) to the multicomponent Sutherland model and derived the spectrum and degeneracy, and discussed the thermodynamics [10]. In spite of its power and simplicity, however, there are still unsolved issues in the ABA. One is about the validity of the asymptotic region in a system with finite size. Another is about the ill-defined phase shift for equal momenta with different spins. Hence an alternative approach is well worth a trial.

In this Letter, we present an elementary theory for deriving all the energy levels of the Sutherland model with arbitrary internal symmetry. The main idea of our method is first to generalize the calculation in Ref. [1] to a modified model for distinguishable particles. After that, we take into account the internal degree of freedom of particles and symmetrize or antisymmetrize the wave function to represent identical particles. The present theory has the following outstanding features. First, our method easily identifies the ground state and gives the explicit form of the wave function for the most general models with SU(ν, μ) supersymmetry. Second, it gives a microscopic derivation of the energy as a functional of the momentum distribution function. Such an expression with the correct account of degeneracy is important especially for the investigation of thermodynamics. Finally, it is also applicable to lattice models such as the multicomponent *t-J* model. Because of these features, this theory should provide a basis for further development such as the investigation of dynamical properties for systems with internal symmetries.

We consider the following model:

$$\mathcal{H} = -\sum_i \frac{\partial^2}{\partial x_i^2} + \frac{2\pi^2}{L^2} \sum_{i<j} \frac{\lambda(\lambda - \zeta M_{ij})}{\sin^2[\pi(x_i - x_j)/L]}, \qquad (1)$$

where M_{ij} is the exchange operator of coordinates of particles i and j [11]. The size of the system is given by L, ζ is 1 or -1, and the dimensionless coupling parameter λ is positive. Since $\mathcal{H}$ does not depend on the internal symmetry of the particles, we consider only the orbital part of the wave function $\Psi(\{x\})$. For the moment, we regard particles as distinguishable and do not impose any permutation properties on the wave functions. Here we write Ψ in the form $\Psi = \Psi_{0,\zeta}\Phi$, where $\Psi_{0,\zeta}$ is the "absolute ground state," by which we mean the ground state in the case where no restrictions to the symmetry of the eigenfunction are imposed. For $\Psi_{0,\zeta}$ we can replace ζM_{ij} in Eq. (1) by 1 since the wave function satisfying $\zeta M_{ij}\Psi_{0,\zeta} = \Psi_{0,\zeta}$ minimizes the repulsion. Then the result of Ref. [1] for the single-component system can be used, and the explicit form of $\Psi_{0,\zeta}$ is given by

$$\Psi_{0,\zeta} = \prod_{i<j} \left| \sin \frac{\pi(x_i - x_j)}{L} \right|^\lambda [\mathrm{sgn}(x_i - x_j)]^{1-\zeta/2}, \qquad (2)$$

with the eigenenergy

$$E_0 = \frac{1}{3}\left(\frac{\lambda\pi}{L}\right)^2 N(N^2 - 1). \qquad (3)$$

Here N is the total number of particles.

Note that $\Psi_{0,\zeta}$ is the totally symmetric (antisymmetric) function when $\zeta = 1(-1)$. Considering Φ as a function

 0031-9007/95/74(7)/1222(4)$06.00

of the variables $z_j = \exp(i2\pi x_j/L)$, we obtain the eigenvalue equation for Φ,

$$\mathcal{H}'\Phi = \left[\sum_j \left(z_j \frac{\partial}{\partial z_j} \right)^2 + \lambda(N-1)\sum_j z_j \frac{\partial}{\partial z_j} + 2\lambda \sum_{j<k} h_{jk} \right]\Phi = \epsilon\Phi, \quad (4)$$

with

$$h_{jk} = \frac{z_j z_k}{z_j - z_k}\left[\left(\frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_k} \right) - \frac{1}{z_j - z_k}(1 - M_{jk}) \right]. \quad (5)$$

The eigenvalue ϵ is related to the original eigenenergy E by $\epsilon = [L/(2\pi)]^2(E - E_0)$.

Since the system has translational symmetry, we consider eigenfunctions with the total momentum Q. We take a complete basis set of plane waves

$$\phi_{\kappa,P} = \prod_i z_{p(i)}^{\kappa_i}, \qquad \sum_i^N \kappa_i = Q, \quad (6)$$

where $P = (p(1), p(2), \ldots, p(N))$ is an Nth order permutation and $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_N)$ is a set of integers ordered so that $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_N$. The action of $\mathcal{H}'$ to $\phi_{\kappa,P}$ gives

$$\mathcal{H}'\phi_{\kappa,P} = \sum_i [\kappa_i^2 + \lambda(N-1)\kappa_i]\phi_{\kappa,P} + 2\lambda \sum_{j<k} h_{p(j)p(k)}\phi_{\kappa,P}. \quad (7)$$

After some calculation, we obtain

$$h_{p(j)p(k)}\phi_{\kappa,P} = -\kappa_k \phi_{\kappa,P} + R(\{z_{p(i)}\}), \quad (8)$$

where R is given by

$$R = \begin{cases} \sum_{l=1}^{\kappa_j - \kappa_k - 1} (\kappa_j - \kappa_k - l)(z_{p(k)} z_{p(j)}^{-1})^l \phi_{\kappa,P}, & \text{if } \kappa_j \geq \kappa_k + 2, \\ 0, & \text{otherwise}. \end{cases} \quad (9)$$

We then define the order of the basis. Let $\kappa' = (\kappa_1', \kappa_2', \ldots, \kappa_N')$ be another set of momenta. We write $\kappa' < \kappa$ if the first nonvanishing difference $\kappa_i - \kappa_i'$ is positive. For instance, $(1,1,1,1,1) < (2,1,1,1,0) < (2,2,1,0,0)$. Similarly, we define the order of the permutation $P = (p(1), p(2), \ldots, p(n))$. If P' is another permutation, we write $P' < P$ if the first nonvanishing difference $p(i) - p'(i)$ is positive. Each function of the basis is characterized by κ and the permutation P. We write $(\kappa', P') < (\kappa, P)$ if $\kappa' < \kappa$, or $\kappa' = \kappa$ and $P' < P$. The off-diagonal elements of $\mathcal{H}'$ come from the second term of Eq. (7). From Eq. (9) we see that $\langle \kappa', P' | \mathcal{H}' | \kappa, P \rangle$ is zero when $(\kappa', P') > (\kappa, P)$. Writing out $\mathcal{H}'$ in the ordered basis, the matrix is an upper triangular one in which all matrix elements below the diagonal vanish. We can obtain all the eigenenergies from the diagonal elements which come from the first terms of the right-hand side of Eqs. (7) and (8) as follows:

$$\epsilon_{\kappa,P} = \epsilon_\kappa = \sum_j [\kappa_j^2 + \lambda(N-1)\kappa_j] - 2\lambda \sum_{j<k} \kappa_k = \sum_j [\kappa_j^2 + \lambda(N + 1 - 2j)\kappa_j]. \quad (10)$$

Here we introduce the momentum distribution function $\nu(k)$ by $\nu(k) = \sum_j \delta(\kappa_j, k)$ with $\delta(\kappa_j, k)$ the Kronecker delta symbol. Then we can rewrite the expression (10) as a functional of $\nu(k)$,

$$\epsilon_\kappa = \sum_{k=-\infty}^{\infty} k^2 \nu(k) + \frac{\lambda}{2} \sum_{k,k'} |k - k'| \nu(k)\nu(k'). \quad (11)$$

All the eigenfunctions are written in the form

$$\Phi_{\kappa,P} = \phi_{\kappa,P} + \sum_{\kappa'<\kappa} \sum_{P'} a_{\kappa',P'} \phi_{\kappa',P'}. \quad (12)$$

Each eigenvalue depends only on κ and is independent of the permutation P. Thus, there is one-to-one correspondence between each eigenstate $\Phi_{\kappa,P}$ of the present model and that of the free system ($\lambda = 0$) given by $\phi_{\kappa,P}$.

Now we consider particles as SU(ν) fermions and calculate the ground state under the given color distribution $\{M_\sigma\}_{\sigma=1}^{\nu}$, where M_σ is the total number of particles with the color σ. First we consider the case $\zeta = 1$. Since the absolute ground state $\Psi_{0,1}$ is totally symmetric, the momenta of particles with the same color are ordered nonequally $(\cdots \kappa_j > \kappa_{j+1} \cdots)$. We introduce the momentum distribution of particles with the color σ by

$$\nu_\sigma(k) = \sum_j \delta(\kappa_j, k)\delta(\sigma, \sigma_j). \quad (13)$$

In this case $\nu_\sigma(k)$ is either 0 or 1. The energy relative to E_0 is given by Eq. (11) with $\nu(k) = \sum_\sigma \nu_\sigma(k)$. Let us consider the following momentum distribution:

$$\nu_\sigma(k) = \theta(M_\sigma/2 - |k|), \quad (14)$$

where $\theta(k)$ is the step function, and each M_σ is taken to be odd. It is clear that Eq. (14) gives the minimum for both the first and second terms in Eq. (11) as SU(ν) fermions. Hence this distribution gives the ground state.

Let κ_g be the set of ordered momenta corresponding to the momentum distribution (14). The ground-state wave function Φ_g is given by antisymmetrization of $\Phi_{\kappa_g,P} P(\{M_\sigma\}) \chi(\{\sigma_j\})$, where χ is a color function and $P(\{\sigma_j\})$ is given by

$$P(\{\sigma_j\}) = \prod_{\sigma=1}^{\nu} \delta\left(M_\sigma, \sum_{j=1}^{N} \delta(\sigma_j, \sigma) \right). \quad (15)$$

First we antisymmetrize the coordinate of particles with the same color. Because of the compact momentum distribution, the only part in $\Phi_{\kappa_g,P}$ that survives antisymmetrization is $\phi_{\kappa_g,P}$. Using the formula for the Vandermonde determinant, the antisymmetrization gives

$$\sum_P (-1)^P \prod_{j=1}^{M_\sigma} z_{P(j)}^{-(M_\sigma+1)/2+j} = \prod_{j=1}^{M_\sigma} z_j^{-(M_\sigma-1)/2} \prod_{j<k}^{M_\sigma} (z_j - z_k), \tag{16}$$

where $(-1)^P$ denotes the sign of the permutation. We must multiply this by the antisymmetric color function for different colors. The result is given by

$$\Phi_g = \prod_{j=1}^{N} z_j^{-(M_{\sigma_j}-1)/2} \prod_{j<k} (z_j - z_k)^{\delta(\sigma_j,\sigma_k)} P(\{M_\sigma\}) \times \exp\left[i\frac{\pi}{2}\mathrm{sgn}(\sigma_j - \sigma_k)\right]. \tag{17}$$

Ha and Haldane proved that this wave function is an eigenfunction of a model equivalent to Eq. (1) with $\zeta = 1$, and conjectured that it provides the ground-state wave function [5]. We have proved here that their conjecture is correct. To see the equivalence between the models we note that $-M_{ij}$ in Eq. (1) can be replaced by the color exchange operator P_{ij} for SU(ν) fermions. On the other hand, if M_σ is even, the ground state is degenerate because the highest momentum occupied can be with either plus or minus sign. Provided that N/ν is an odd integer, the minimum energy distribution among all the color distributions $\{M_\sigma\}$ is given by $M_\sigma = N/\nu$ for all σ's.

Next we consider the case $\zeta = -1$ for SU(ν) fermions. In this case the absolute ground state $\Psi_{0,-1}$ is totally antisymmetric and the momentum distribution is bosonic, $\nu_\sigma(k) = 0, 1, 2, \ldots$. The ground state is given by $\nu_\sigma(k) = M_\sigma \delta(k,0)$ for all σ's and the ground-state energy is just E_0. The corresponding eigenfunction is $\Psi = \Psi_{0,-1}(\{z_j\})P(\{\sigma_j\})$. Any distribution $\{M_\sigma\}$ leads to the same energy given by Eq. (3). Hence there is a degeneracy $(N + \nu - 1)!/[(\nu - 1)!\,N!]$ in the ground state.

In the cases where particles are SU(ν) bosons or a mixture of bosons and fermions, the model given by Eq. (1) can be solved in a similar way. Thus we consider the most general SU(ν, μ) model, i.e., particles consisting of SU(μ) fermions and SU(ν) bosons. The wave function for the ground state has not yet been reported for this model. In this case, we can rewrite M_{ij} as

$$M_{ij} = \tilde{P}_{ij} \equiv \begin{cases} -P_{ij} & \text{if both } i\text{th and } j\text{th particles are fermions}, \\ P_{ij} & \text{otherwise.} \end{cases} \tag{18}$$

We first consider the case $\zeta = 1$. The absolute ground state is given by

$$\Psi_{0,1} = \prod_{j<k}^{N^B} |\xi_j - \xi_k|^\lambda \prod_{j<k}^{N^F} |\omega_j - \omega_k|^\lambda \prod_j^{N^B}\prod_k^{N^F} |\xi_j - \omega_k|^\lambda, \tag{19}$$

where $\xi_j(\omega_j)$ and σ_j^B (σ_j^F) are variables for the complex coordinate and the color of the jth boson (fermion), respectively, and N^B (N^F) is the total number of bosons (fermions).

Let the momentum distribution for bosons (fermions) with the color α (β) be given by $\{\nu_\alpha^B(k)\}$ [$\{\nu_\beta^F(k)\}$], and the color distribution by $\{M_\alpha^B\}_{\alpha=1}^\nu$ [$\{M_\beta^F\}_{\beta=1}^\mu$], where each M_β^F is taken to be odd. Since the absolute ground state (19) is symmetric, the $\nu_\alpha^B(k)$'s take all non-negative integers $0, 1, 2, \ldots$ and the $\nu_\beta^F(k)$'s are 0 or 1. The minimum energy distribution is given by

$$\nu_\alpha^B(k) = M_\alpha^B \delta(k,0), \quad \nu_\beta^F(k) = \theta(M_\beta^F/2 - |k|). \tag{20}$$

The corresponding wave function Ψ is given by $\Psi_{0,1}\Phi_g$, where

$$\Phi_g = \prod_j^{N^F} \omega_j^{-K(\sigma_j^F)} \prod_{j<k}^{N^F} (\omega_j - \omega_k)^{\delta(\sigma_j^F,\sigma_k^F)} \times \exp\left[\frac{i\pi}{2}\mathrm{sgn}(\sigma_j^F - \sigma_k^F)\right] P(\{M_\alpha^B\})P(\{M_\beta^F\}), \tag{21}$$

with $K(\sigma) = (M_\sigma^{F,B} - 1)/2$. Similarly, the ground state in the case $\zeta = -1$ is given by $\Psi_{0,-1}\Phi_g'$, where

$$\Phi_g' = \prod_j^{N^B} \xi_j^{-K(\sigma_j^B)} \prod_{j<k}^{N^B} (\xi_j - \xi_k)^{\delta(\sigma_j^B,\sigma_k^B)} \times \exp\left[\frac{i\pi}{2}\mathrm{sgn}(\sigma_j^B - \sigma_k^B)\right] P(\{M_\alpha^B\})P(\{M_\beta^F\}). \tag{22}$$

Our method is also applicable to lattice models by taking the limit $\lambda \to \infty$ [10,12,13]. In the strong coupling limit, a part of the potential $\lambda^2 \sum_{j<k} \sin^{-2}[\pi(x_j - x_k)/L]$ enforces particles to localize with a lattice spacing L/N. Up to order $O(\lambda)$, the Hamiltonian decouples into

$$\mathcal{H} = \mathcal{H}_{\mathrm{ela}} + \mathcal{H}_{\mathrm{lat}} + E_{\mathrm{Mad}}, \tag{23}$$

where $\mathcal{H}_{\mathrm{ela}}$, $\mathcal{H}_{\mathrm{lat}}$, and E_{Mad} are the elastic and lattice Hamiltonians and the Madelung energy, respectively. We define u_j as the displacement from the jth lattice point ($j = 1, 2, \ldots, N$) and introduce a function

$$\tilde{d}(j) = 3\sin^{-4}\left(\frac{\pi j}{N}\right) - 2\sin^{-2}\left(\frac{\pi j}{N}\right). \tag{24}$$

Then we obtain

$$\mathcal{H}_{\mathrm{ela}} = -\sum_j \frac{\partial^2}{\partial u_j^2} + 2\lambda^2\left(\frac{\pi}{L}\right)^4 \sum_j \left[\sum_{k(\neq j)} \tilde{d}(j-k)\right] u_j^2 - 4\lambda^2\left(\frac{\pi}{L}\right)^4 \sum_{j<k} \tilde{d}(j-k) u_j u_k, \tag{25}$$

where j and k are no longer indices of particles but those of sites. We rewrite $\mathcal{H}_{\rm ela}$ as

$$\mathcal{H}_{\rm ela} = \sum_{q=0}^{N-1} 4\lambda\left(\frac{\pi}{L}\right)^2 q(N-q)\left(\hat{b}_q^\dagger \hat{b}_q + \frac{1}{2}\right), \tag{26}$$

where $\hat{b}_q^\dagger$ ($\hat{b}_q$) is the creation (annihilation) operator of the phonon field. In deriving the expression (26), we have used the following result:

$$\sum_{l=1}^{N-1} \tilde{d}(l) \exp\left(\frac{i2\pi ql}{N}\right) = -2q^2(q-N)^2 + \frac{N^4-1}{15}. \tag{27}$$

The rest of $\mathcal{H}$ is given by

$$E_{\rm Mad} = 2\left(\frac{\pi}{L}\right)^2 \sum_{j<k} \frac{\lambda^2}{\sin^2[\pi(j-k)/N]} = \frac{1}{3}\left(\frac{\pi\lambda}{L}\right)^2 N(N^2-1), \tag{28}$$

$$\mathcal{H}_{\rm lat} = -2\left(\frac{\pi}{L}\right)^2 \sum_{j<k} \frac{\lambda\zeta\tilde{P}_{jk}}{\sin^2[\pi(j-k)/N]}, \tag{29}$$

where $\tilde{P}_{jk}$ is the color exchange operator between the jth and kth sites. When $(\zeta,\nu,\mu) = (1,0,2)$ or $(-1,2,0)$, $\mathcal{H}_{\rm lat}$ becomes the Hamiltonian of the HS model. In the case of $(\zeta,\nu,\mu) = (1,1,2)$ or $(-1,2,1)$, it becomes the Hamiltonian of the supersymmetric t-J model.

We now derive the wave function $\Psi_g = \Psi_{0,1}\Phi_g$ of the ground state with $\zeta = 1$ in the limit of $\lambda \to \infty$. The absolute ground state $\Psi_{0,1}$ becomes

$$\Psi_{0,1} \to \exp\left[-\lambda\left(\frac{\pi}{L}\right)^2 \sum_{q=0}^{N-1} q(N-q)u(q)u(N-q)\right] \times (\text{numerical factor}), \tag{30}$$

where $u(q)$ is the Fourier transform of the displacement u_j. The $\Psi_{0,1}$ corresponds to the zero-point motion of the phonon field. Factoring out the phonon part, we obtain the ground state of the SU(ν,μ) lattice model, which is nothing but Φ_g given by Eq. (21). The remaining effect of $\Psi_{0,1}$ is the exclusion of multiple occupation of each site. In the special case of SU(1,2), it is therefore proved that the Gutzwiller wave function is indeed the ground state of the t-J model [4].

Thus the present theory gives unified treatment of the family of Sutherland-type models. In every case, our theory derives all the energy levels by a simple calculation. We have proved in this way the previous conjectures on the ground states. Furthermore, the simplicity of the method permits us to derive new results for the ground-state wave function of SU(ν,μ) models in both the continuum space and the lattice.

Lastly, we make a brief remark on another application of our theory. A similar approach is useful for the Calogero model with internal symmetry as well [14]. As a matter of fact, the Calogero model is much more tractable than the Sutherland model. For the Calogero model, the local operators for energy boost were already found [11,15]. By using such operators, we have obtained all the eigenstates of the SU(ν) Calogero model and proved that the eigenfunction proposed in Refs. [7,14] is the exact ground state. The details will be discussed elsewhere.

[1] B. Sutherland, J. Math. Phys. (N.Y.) **12**, 246 (1971); **12**, 251 (1971); Phys. Rev. A **4**, 2019 (1971); **5**, 1372 (1972).
[2] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988).
[3] B. S. Shastry, Phys. Rev. Lett. **60**, 639 (1988).
[4] Y. Kuramoto and H. Yokoyama, Phys. Rev. Lett. **67**, 1338 (1991).
[5] Z. N. C. Ha and F. D. M. Haldane, Phys. Rev. B **46**, 9359 (1992).
[6] N. Kawakami, Phys. Rev. B **46**, 3191 (1992).
[7] J. A. Minahan and A. P. Polychronakos, Phys. Lett. B **302**, 265 (1993).
[8] F. D. M. Haldane, Z. N. C. Ha, J. C. Talstra, D. Bernard, and V. Pasquier, Phys. Rev. Lett. **69**, 2021 (1992).
[9] D. Bernard, M. Gaudin, F. D. M. Haldane, and V. Pasquier, J. Phys. A **26**, 5219 (1993).
[10] B. Sutherland and B. S. Shastry, Phys. Rev. Lett. **71**, 5 (1993).
[11] A. P. Polychronakos, Phys. Rev. Lett. **69**, 703 (1992).
[12] A. P. Polychronakos, Phys. Rev. Lett. **70**, 2329 (1993).
[13] F. D. M. Haldane, in *Correlation Effects in Low-Dimensional Electron Systems*, edited by A. Okiji and N. Kawakami (Springer-Verlag, Berlin, 1994), p. 3.
[14] K. Vacek, A. Okiji, and N. Kawakami, J. Phys. Soc. Jpn. **62**, 3818 (1993).
[15] L. Brink, T. H. Hanson, and M. A. Vasiliev, Phys. Lett. B **286**, 109 (1992).