Electric-Field-Induced Nonlinear Bloch Oscillations and Dynamical Localization

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The dynamics of a nonlinear Schrödinger chain in a time-varying, spatially uniform electric field is studied and proven to be integrable. In the limit of a static electric field, the system exhibits a periodic evolution which is a nonlinear counterpart of Bloch oscillations. It is shown that localization can be dynamically induced by a temporally harmonic field as a consequence of parametric resonances at certain field strengths. The effects of integrability-breaking discrete lattice terms are studied numerically: Nonlinear Bloch oscillations and dynamical localization are found to be a property of the lattice and not limited to the integrable case.

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The question of how a charged particle will behave in a perfectly periodic crystal in the presence of a static electric field has a long and fascinating history [1]. The advent of technology for fabricating semiconductor superlattices has stirred new interest in the implications of this basic quantum transport problem. For example, one potential application is the realization of a Bloch oscillator as a fast emitter of electromagnetic radiation. It is known that an electric field of strength $\mathcal{E}_0$ will induce localization of the particle, which executes a "Bloch oscillation" with the Bloch frequency $\omega_B = e\mathcal{E}_0 a/\hbar$. Associated with this phenomenon is the Stark ladder (SL), $n(e\mathcal{E}_0 a)$ [2], where n is an integer and a is the lattice spacing which here will be set to unity (we also set $\hbar = 1$, $e = 1$ in the following). The periodic oscillatory motion of the particle reflects the fact that the SL, i.e., the energy eigenspectrum of the system, is evenly spaced. Recently, these phenomena have been confirmed by various laboratory experiments [3]. Since the Bloch oscillation is a consequence of the underlying periodic lattice structure, the essential physics should remain the same with the periodic crystal being replaced by a semiconductor superlattice, an array of coupled quantum wells or a molecular chain. A simple theoretical approach which captures the underlying physics is to use tight-binding models. In this context, dynamical localization of a charged particle induced by a *time-periodic* electric field has also been discussed [4].

In the present Letter we consider a generalization of the tight-binding model to a *nonlinear* case where the nonlinearity is an effective self-interaction of the particle. This nonlinearity can be induced by, e.g., interaction with the lattice as in the case of excitons in a molecular chain [5], or electron excitations in polaronic contexts [6]. The nonlinear system without the external electric field is a discrete, one-dimensional nonlinear Schrödinger equation (NLS) for which nonlinear localized excitations have been extensively studied [7,8]: The role of nonintegrability as a perturbation to an integrable system has been examined [8,9], also novel discreteness effects have been contrasted with the continuum limit [8]. In the following we will show that the system, in a certain limit and in the presence of a time-dependent electric field, is integrable and possesses an infinite number of conserved quantities. For the temporally harmonic field, there exist resonance conditions for which the evolution of the wave function of the particle is periodic and the system exhibits dynamical localization. Furthermore, as the frequency of the external field approaches zero, we recover the discrete NLS in a static electric field [10], and the periodic motion persists with precisely the Bloch frequency $\mathcal{E}_0$ which occurs without the nonlinearity. We note that these similarities with the linear problem show that, for certain nonlinearities, some linear characteristics persist and can be extended to highly nonlinear regimes. Mathematically, this is related to the fact that the time evolution of a system for which the inverse scattering transform (IST) is applicable is determined by an associated linear problem. Finally, we emphasize that in both the linear and nonlinear cases Bloch oscillation and dynamical localization are consequences of the underlying periodic structure, i.e., lattice discreteness: We will show an example of a discrete nonlinear system which is not integrable [8,11], but for which the above phenomena nonetheless substantially persist.

The governing differential-difference equation for our chain system is

$$i\dot{\psi}_n = -(\psi_{n+1} + \psi_{n-1}) - \mu(\psi_{n+1} + \psi_{n-1})|\psi_n|^2 - 2\nu\psi_n|\psi_n|^2 + V_n\psi_n\,, \tag{1}$$

where the overdot denotes the time derivative, n is a site index, and V_n is an external field. The μ term and ν term can be viewed as the first order correction to the intersite overlap integral and the on-site frequency, respectively, taking into account the nonlinearity as the induced self-interaction. Here time and space have been scaled such that the zeroth order hopping constant is unity.

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By a simple gauge transformation the zeroth order on-site frequency has been shifted to zero. The corresponding Hamiltonian can be written as

$$H = -\sum_n (\psi_n \psi_{n+1}^* + \psi_n^* \psi_{n+1}) - \frac{2\nu}{\mu}\sum_n |\psi_n|^2 + \frac{1}{\mu}\sum_n \left(\frac{2\nu}{\mu} + V_n\right) \ln(1 + \mu|\psi_n|^2), \quad (2)$$

with the deformed Poisson brackets $\{\psi_n, \psi_m\} = 0 = \{\psi_n^*, \psi_m^*\}$, $\{\psi_n, \psi_m^*\} = i(1 + \mu|\psi_n|^2)\delta_{nm}$.

The system (1) has been considered in Ref. [8] in the absence of the external potential. In the limit $\nu = 0$, $V_n = 0$, it is the Ablowitz-Ladik lattice system [12], which is completely integrable and possesses an infinite number of conservation laws for the infinite lattice. In general, the system (1) is not integrable. When the external potential is time dependent, the system is no longer conservative. Nevertheless, there exists at least one conserved quantity, $\mathcal{N} = \mu^{-1} \sum_n \ln(1 + \mu|\psi_n|^2)$, which serves as a norm of the system as long as V_n is real. In this Letter, we study the system with a potential in the form

$$V_n = \mathcal{E}(t)\, n\,, \quad (3)$$

where $\mathcal{E}(t)$ is *any* function of time. This potential corresponds to a time-dependent, spatially uniform electric field along the chain direction. It generalizes the case of a static linear potential, whose soliton dynamics was studied in Ref. [10], and for which we will comment on below regarding the dynamics of the radiation as well as that of solitons.

In what follows, unless specified otherwise, we will restrict ourselves to the system with $\nu = 0$ and the potential (3) for which it can be shown that Eq. (1) is exactly integrable (as in usual IST analyses, we assume that the wave function decays sufficiently rapidly at boundaries). The general case with $\nu \neq 0$ can be treated perturbatively and a detailed study will be given elsewhere [13]. The system (1) with $\nu = 0$ and $V_n = \mathcal{E}(t)\, n$ admits the zero-curvature representation [14]:

$$\dot{L}_n + L_n A_n - A_{n+1} L_n = 0\,, \quad (4)$$

where

$$L_n = \begin{pmatrix} \lambda & i\psi_n^* \\ i\psi_n & \lambda^{-1} \end{pmatrix},$$

$$A_n = \begin{pmatrix} i(1-\psi_n^*\psi_{n-1}-\lambda^2+f_n) & -\lambda^{-1}\psi_{n-1}^*+\lambda\psi_n^* \\ -\lambda^{-1}\psi_n+\lambda\psi_{n-1} & -i(1-\psi_n\psi_{n-1}^*-\lambda^{-2}+f_n) \end{pmatrix},$$

with $f_n = \mathcal{E}(t)n/2 + \delta$, $\delta = -\mathcal{E}(t)/4 - 1$ [where we have set $\mu = 1$ by virtue of the scaling property in Eq. (1)] [15]. We note that the spectral parameter for the above IST problem is time dependent,

$$\lambda = \lambda_0 \Lambda(t), \qquad \Lambda(t) \equiv \exp\left[i \int_0^t \frac{\mathcal{E}(\tau)}{2}\, d\tau \right], \quad (5)$$

and therefore requires slight modifications to the usual time-independent spectral theory of IST. It can be solved by generalizing the approaches formulated in Refs. [16,17] to the lattice. It can be shown that the scattering data obey the following time evolution:

$$a[\lambda(t), t] = a(\lambda(0), 0)\,, \quad \frac{b}{a}[\lambda(t), t] = p\left(\frac{b}{a}\right)_0 S(t)\,,$$
$$c_j(t) = c_{j0}\, \Lambda(t) S(t)|_{\lambda_j}\,, \quad (6)$$
$$S(t) \equiv \exp\left[-i \int_0^t d\tau\, (2 + 2\delta - \lambda^2 - \lambda^{-2}) \right],$$

where λ_j is the jth zero of $a(\lambda, t)$ and $\bar{a} = a^*$, $\bar{b} = b^*$, $\bar{\lambda}_j = (\lambda_j^*)^{-1}$, and $\bar{c}_j = c_j^*/(\lambda_j^*)^2$, due to the well-known symmetry [12] between the off-diagonal elements in the scattering matrix L_n. The system still possesses an infinite number of constants of motion which are of the form, $\tilde{C}_m = \Lambda(t)^{2m} C_m$, $m = 1, 2, \ldots$, where C_m's have the same algebraic expressions as those for the conserved quantities in the absence of V_n, (see, e.g., [12]). For example, the first in the hierarchy is $C_1 = -\sum_{-\infty}^{+\infty} \psi_n^* \psi_{n+1}$, whose real and imaginary parts are related to, respectively, the energy and momentum of the system and are conserved when $V_n = 0$. For the present case, C_1 has the temporal behavior $\Lambda(t)^{-2}\tilde{C}_1$, where $\tilde{C}_1$ is a constant. Note that the norm $\mathcal{N}$ is a conserved quantity in addition to the conserved hierarchy.

We now particularize to the temporally harmonic potential $\mathcal{E} = \mathcal{E}_0 \cos\omega t$, for which we have the periodic flow of the spectral parameter, $\Lambda(t) = \exp[i(\mathcal{E}_0 \sin\omega t)/2\omega]$, and

$$S(t) = \Lambda(t) \exp[-(\lambda_0^2 - \lambda_0^{-2})v(t) + i(\lambda_0^2 + \lambda_0^{-2})u(t)]\,, \quad (7)$$

where

$$u(t) = J_0\left(\frac{\mathcal{E}_0}{\omega}\right) t + \frac{1}{\omega}\sum_{n=1}^{\infty} \frac{1}{n} J_{2n}\left(\frac{\mathcal{E}_0}{\omega}\right) \sin(2n\omega t)\,,$$
$$v(t) = \frac{2}{\omega}\sum_{n=1}^{\infty} \frac{1}{2n-1} J_{2n-1}\left(\frac{\mathcal{E}_0}{\omega}\right) \times \{1 - \cos[(2n-1)\omega t]\}, \quad (8)$$

and $J_n(x)$ are Bessel functions of the first kind. It can be readily seen that the scattering data are periodic with the period $T = 2\pi/\omega$, if $\mathcal{E}_0/\omega = x_i^0$, $i = 1, 2, \ldots$, with x_i^0 the ith root of the zeroth order Bessel function $J_0(x)$. It then follows by IST that the time evolution of the system (1) with $\nu = 0$ is also periodic with the same period as the external driving. We shall refer to the series of values of the ratio $\mathcal{E}_0/\omega$ as parametric "resonance conditions." Physically, these lead to the recurrence of the field in time by the time-periodic potential. In other words, a particle can be *dynamically* trapped by the potential if the particle is initially localized in space. This is precisely the same dynamical localization phenomenon studied in Ref. [4] for *linear* tight-binding chains. For the linear system, there exist the same resonance conditions which must hold in order for the system to exhibit dynamical localization. At resonance, since the time evolution of the system only contains the harmonics of ω, for any initial condition, it is plausible that these harmonics could be experimentally detected when the electric field strength is tuned to the resonance ratios.

In Fig. 1 the evolution of the system (1) with $\nu = 0$ is shown when the resonance condition holds [18]. The recurrence phenomenon is evident. Initially, the wave function was taken as a combination of a wave packet of a Kronecker δ-function form and a soliton wave packet. The wave packet initially localized at a single lattice site gradually disperses into low-amplitude radiation modes and eventually refocuses to the δ function, whereas the soliton never disperses, is dressed by the radiation, and may emerge from the radiation unchanged. We note in passing that this recurrence has a parametric resonance nature [19]. Therefore, any small deviation from the resonance ratio will grow linearly in time; hence there is a slow destruction of the recurrence in time. For example, in Fig. 2 we show an off-resonance example of the evolution starting with the same initial field profile. Here we see dispersion of the radiation modes in contrast to the robust particlelike motion of the soliton.

In the large ω limit, the dynamics of the solitons closely resembles that of solitons without the external potential. This is simply a manifestation in the solitonic dynamics of the Kapitza decomposition of the motion of a particle under a periodic forcing into slow and fast components [19].

We note that the behavior of the system (1) in the $\nu = 0$ integrable limit is qualitatively preserved even in the presence of the on-site nonintegrability, i.e., $\nu \neq 0$. Thus, Fig. 3 displays similar radiation modes to those in Fig. 1, although the soliton slowly disperses, merging into the radiation background for the system with $\mu = 1$, $\nu = -0.5$, and $\mathcal{E}_0/\omega = x_1^0$. It is not surprising that the radiation still exhibits the recurrence qualitatively because, for the low-amplitude wave function, the features of the system can be captured in the linear limit for which the resonance condition for dynamical localization is again exact [4]. The evolution has a qualitative "localized chaos" structure, but is bounded by virtue of the lattice discreteness.

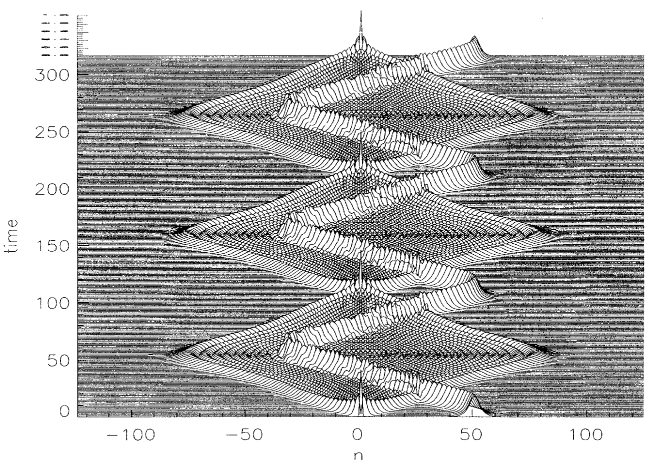

FIG. 1. The evolution of the system (1) with $\mu = 1$ and $\nu = 0$ at the resonance condition $\mathcal{E}_0/\omega = x_1^0$, where x_1^0 is the first zero of the Bessel function $J_0(x)$ and $\omega = 0.06$. Plotted here is $|\psi_n(t)|$ (see text). For all simulations presented here, we used sufficiently long chains that the boundary effects are negligible.

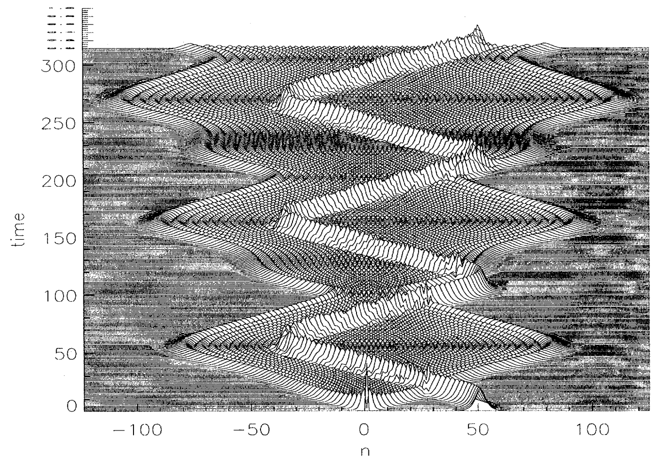

FIG. 2. Example of an off-resonance condition. The parameters are the same as Fig. 1 except that $\mathcal{E}_0/\omega = 0.9x_1^0$.

For the static potential, $V_n = \mathcal{E}_0 n$, we obtain

$$u(t) = \frac{1}{\mathcal{E}_0}\sin(\mathcal{E}_0 t),$$
$$v(t) = \frac{1}{\mathcal{E}_0}[1 - \cos(\mathcal{E}_0 t)], \quad (9)$$

which can also be derived by taking $\omega \rightarrow 0$ in Eqs. (8). Again, it can be shown that the system evolves periodically for any initially localized profile. The periodic soliton motion obtained in Ref. [10] serves as a special case here. The temporal period now is $T = 2\pi/\mathcal{E}_0$, which is determined by the strength of the static external field. There are only the harmonics of the

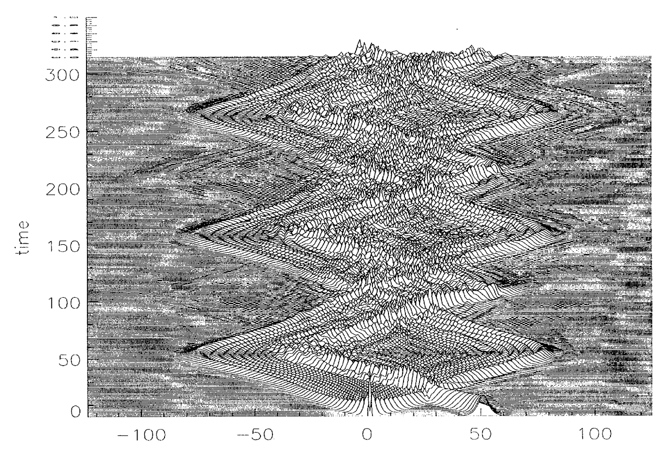

FIG. 3. Dynamics of a nonintegrable system with $\mu = 1$ and $\nu = -0.5$ at $\mathcal{E}_0/\omega = x_1^0$, $\omega = 0.06$ (cf. Fig. 1).

frequency $\mathcal{E}_0$ in the evolution of the system. These harmonics are directly connected to the harmonics of the linear Bloch oscillation in the weakly nonlinear, i.e., low-amplitude, regime, as the IST approaches the Fourier transform in this regime [12]. Hence, this spectrum can be regarded as a generalization of the linear Stark ladder spectrum. Evidently, these characteristics of the evolution as the nonlinear counterpart of Bloch oscillations preserve the features of Bloch oscillations in the linear system. Notice that in the small $\mathcal{E}_0 \ll 1$ limit, only on the time scale $t \ll 1/\sqrt{\mathcal{E}_0}$ can we recover the evolution of the system for the potential-free case. In general, no matter how small $\mathcal{E}_0$ is, the periodic motion persists. This is true of both the linear and the nonlinear cases. It is worthwhile comparing with the evolution of a wave packet for the continuum limit in the presence of a spatially linear, static potential. For the linear Schrödinger equation, the widthof the wave packet increases, whereas the solitonic wave packet for the nonlinear Schrödinger equation has a permanent wave profile. However, both of their centers execute a parabolic motion in time, just as for a classical particle moving in a constant gravitational field [17]. There is no periodic motion. Obviously, when discretization is introduced in numerical simulations for, e.g., these systems, care should be taken about the time scale on which the results of the simulations are a valid representation of the continuum limit.

In summary, we have established that the NLS, Eq. (1), with $\nu = 0$, is integrable in an external time-varying, spatially uniform electric field. This work further extends our understanding of dynamics in discrete nonlinear systems under external driving, and is not restricted to simple soliton dynamics. We have demonstrated that the discrete nonlinear system considered still exhibits the phenomena of Bloch oscillation and dynamical localization, as a generalization of the known linear cases. For the nonintegrable situation, these phenomena also qualitatively persist. We thus suggest that these phenomena are fundamental to discrete lattices and may be observable in appropriate experimental settings even where the nonlinearity is strong.

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Note added.—After the submission of this Letter, we became aware that the special case of $\mathcal{E}(t) = \text{const}$ was considered originally in Bruschi *et al.* [20], and the integrability of the general case was noted in Konotop *et al.* [21].

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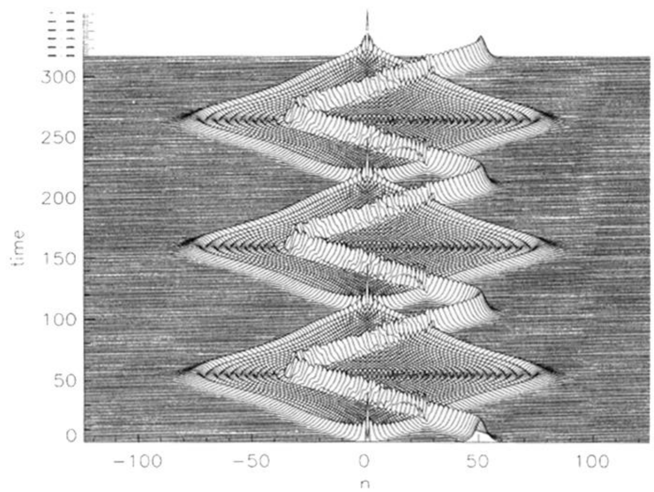

FIG. 1. The evolution of the system (1) with $\mu = 1$ and $\nu = 0$ at the resonance condition $\mathcal{E}_0/\omega = x_1^0$, where x_1^0 is the first zero of the Bessel function $J_0(x)$ and $\omega = 0.06$. Plotted here is $|\psi_n(t)|$ (see text). For all simulations presented here, we used sufficiently long chains that the boundary effects are negligible.

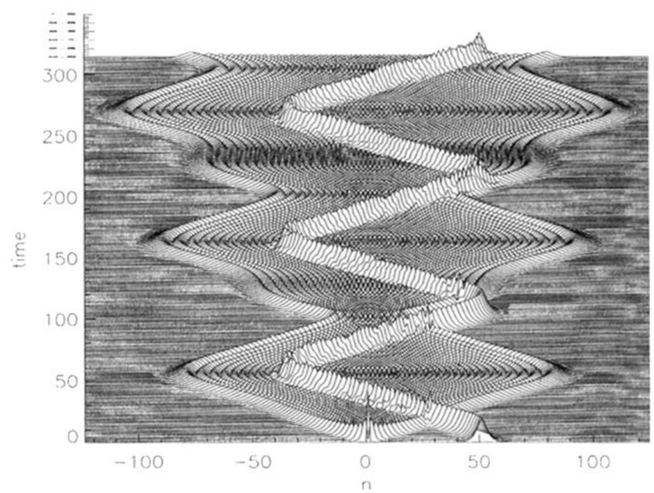

FIG. 2. Example of an off-resonance condition. The parameters are the same as Fig. 1 except that $\mathcal{E}_0/\omega = 0.9x_1^0$.

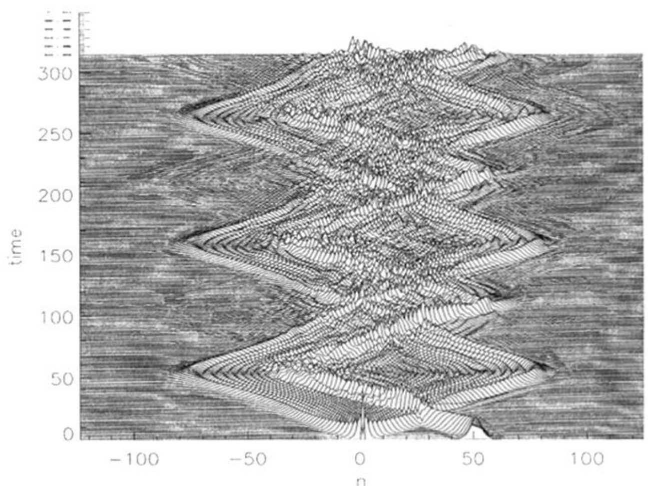

FIG. 3. Dynamics of a nonintegrable system with $\mu = 1$ and $\nu = -0.5$ at $\mathcal{E}_0/\omega = x_1^0$, $\omega = 0.06$ (cf. Fig. 1).